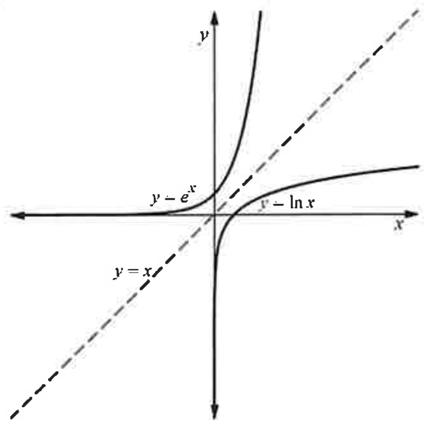
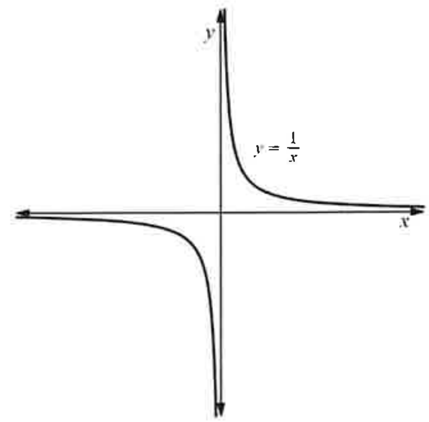
# 10.3 The Natural Logarithm as an Area

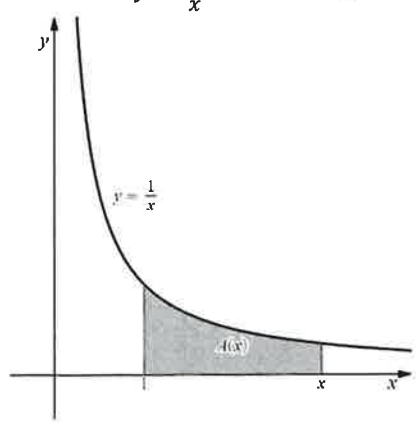
The function $y = \ln x$ is defined as the **inverse** of the function $y = e^x$. Recall that the derivative of the natural logarithm is $\frac{d}{dx}(\ln x) = \frac{1}{x}$. The function $y = \frac{1}{x}$ is continuous and positive in the interval $(0, \infty)$, and continuous and negative in the interval $(0. -\infty)$. The graphs of these functions are shown below.




Notice that the graphs of $y = \ln x$ and $y = e^x$ are reflections of each other around the line $y = x$. This is true of all inverse functions.

Find the area under $y = \frac{1}{x}$ from 1 to $x$.




The diagram to the left shows the area $A(x)$ that we are trying to calculate. Since the derivative of $A(x)$ is equal to the function we have the following.

$$A'(x) = \frac{1}{x}$$

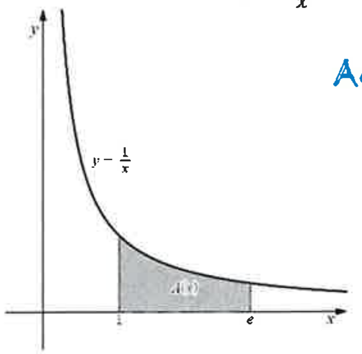
Choosing the antiderivative $F(x) = \ln x$ we can find the area function.

$$\begin{aligned} A(x) &= F(x) - F(1) \\ &= \ln x - \ln 1 \quad \leftarrow \ln 1 = 0 \\ &\boxed{= \ln x} \end{aligned}$$

> If $x > 1$, the natural logarithm $\ln x$ is the area under the curve from $y = \frac{1}{x}$ from 1 to $x$.

**Ex. 1**

Find the area under $y = \frac{1}{x}$ from $x = 1$ to $x = e$.

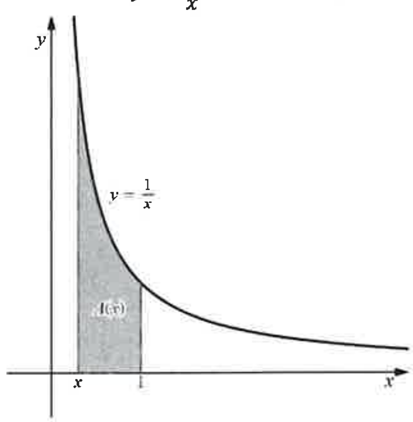



$A(x) = \ln e - \ln 1$

$= 1 - 0$

$= 1$

Find the area under $y = \frac{1}{x}$ from $x$ to 1, $0 < x < 1$.




The diagram to the left shows the area $A(x)$ that we are trying to calculate. Again, since the derivative of $A(x)$ is equal to the function we have the following.

$$A'(x) = \frac{1}{x}$$

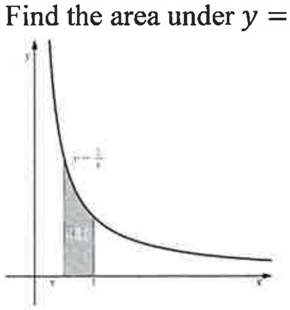
Choosing the antiderivative $F(x) = \ln x$ we can find the area function.

$$\begin{aligned} A(x) &= F(1) - F(x) \\ &= \ln 1 - \ln x \\ &\boxed{= -\ln x} \end{aligned}$$

> The natural logarithm $\ln x$ is the negative of the area under the curve $y = \frac{1}{x}$ from $x$ to 1 for $0 < x < 1$

**Ex. 2**

Find the area under $y = \frac{1}{x}$ from $x = 0.5$ to $x = 1$.



$A(x) = \ln 1 - \ln 0.5$

$= 0 - \ln 0.5$

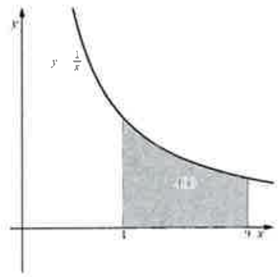
$= -\ln 0.5$

$\doteq 0.693147$

**Ex. 3**

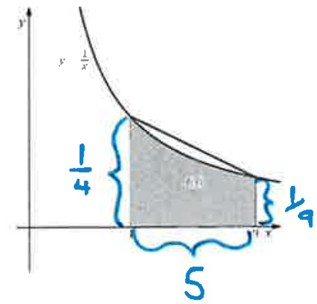
(a) Prove that the area of the region $\{r = \{(x, y)|4 \le x \le 9, 0 \le xy \le 1\}$ is $2\ln 1.5$.
(b) Use an area to prove that $\ln 1.5 < \frac{65}{144}$.

(a)



$A_{(a)} = \ln 9 - \ln 4$
$= \ln(9/4)$
$= \ln 2.25$
$= \ln(1.5)^2$
$= 2\ln 1.5$

(b)



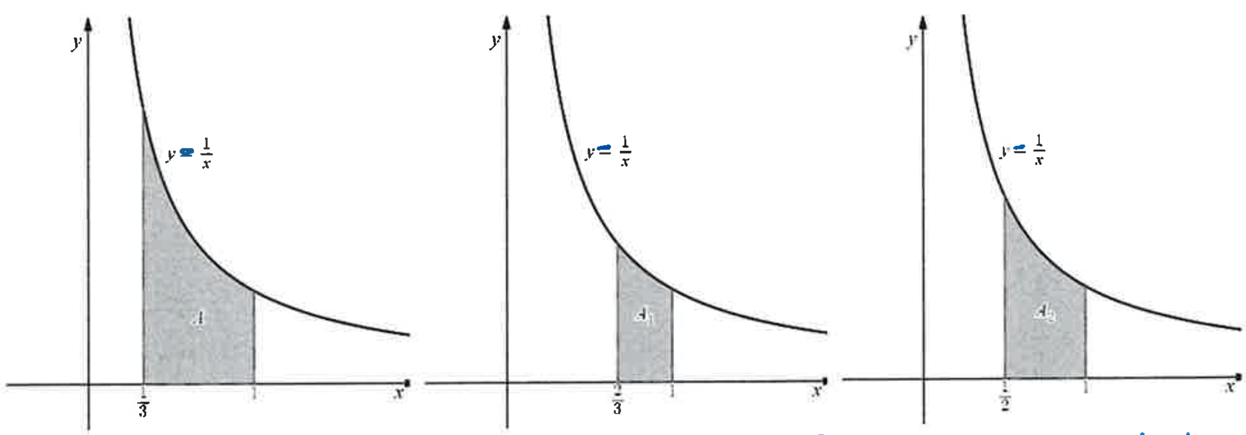

$A = \frac{1}{2}h\left(\frac{1}{9}+\frac{1}{4}\right)$
$= \frac{1}{2}(5)\left(\frac{4+9}{36}\right) = \frac{5}{2}\left(\frac{13}{36}\right)$
$= \frac{65}{72}$

So $2\ln 1.5 < \frac{65}{72}$

$\boxed{\ln 1.5 < \frac{65}{144}}$

**Ex. 4**

Prove that $A = A_1 + A_2$ if $A$, $A_1$, and $A_2$ are the areas in the diagrams shown below.



$A_{(x)} = \ln 1 - \ln\frac{1}{3}$
$= 0 - \ln\frac{1}{3}$
$= -\ln\frac{1}{3}$

$A_{1(x)} = \ln 1 - \ln 2/3$
$= 0 - \ln 2/3$
$= -\ln 2/3$

$A_{2(x)} = \ln 1 - \ln\frac{1}{2}$
$= 0 - \ln\frac{1}{2}$
$= -\ln\frac{1}{2}$

$A_1 + A_2 = -\ln 2/3 + -\ln 1/2$
$= -(\ln 2/3 + \ln 1/2)$
$= -\ln 2/3 \cdot 1/2 \rightarrow -\ln\frac{1}{3}$

**Homework Assignment**

- Practice Problems: #1, 2, 3