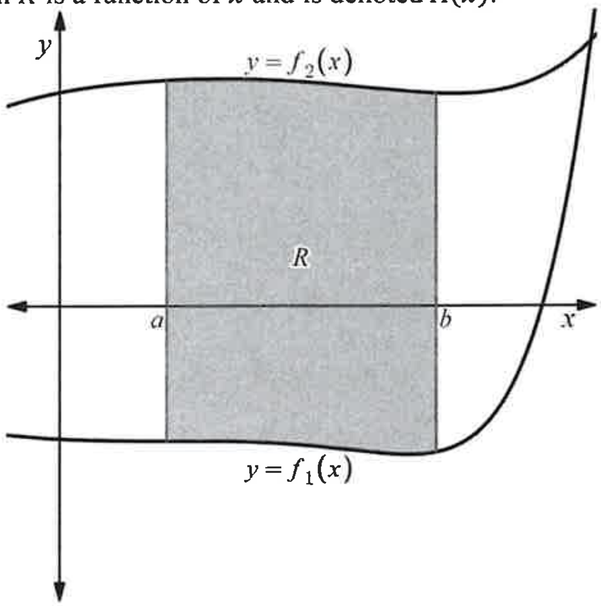
10.2 Area Between Curves

Now we consider a more general area problem. How to find the area of a region between two curves over a particular interval. Region R in the diagram below is between $y = f_1(x)$ and $y = f_2(x)$ from a to x. If a is fixed, the area of region R is a function of x and is denoted $A(x)$.



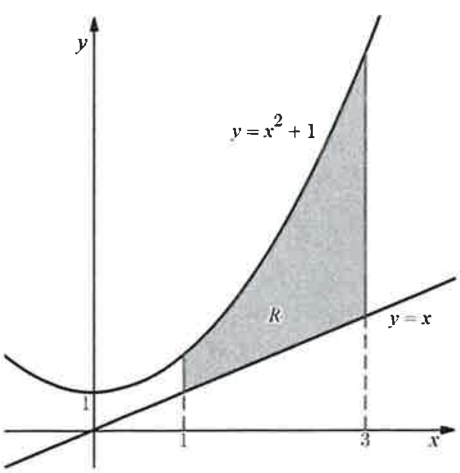
Note that in the interval $[a, x]$, $f_2(x) > f_1(x)$ and it follows like in the previous section that the following must be true.

$$A'(x) = f_2(x) - f_1(x), f_2(x) - f_1(x) \geq 0$$
$$A'(x) = f(x), \text{ where } f(x) = f_2(x) - f_1(x)$$

The problem of finding the area between two curves can be reduced to finding the area under a new function $f(x)$ where $f(x) = f_2(x) - f_1(x)$. In general, if f_1 and f_2 are continuous functions in the interval $[a, b]$, the area between f_1 and f_2 from a to b is the area under $f_2 - f_1$ from a to b.

Ex. 1

Find the area between $y = x^2 + 1$ and $y = x$ from $x = 1$ to $x = 3$.



$y = x^2+1$ is always above $y = x$

so $A'(x) = x^2+1-x$

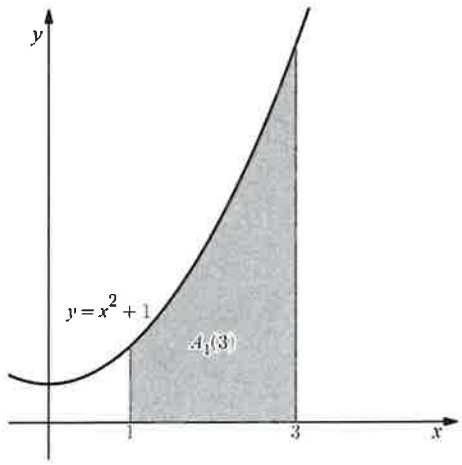
therefore: $A(x) = \frac{1}{3}x^3 + x - \frac{1}{2}x^2$

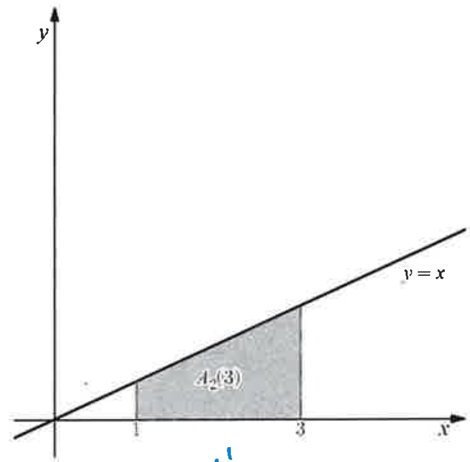
$A(3) = F(3) - F(1)$

$= \frac{1}{3}(3)^3 - \frac{1}{2}(3)^2 + 3 - \left[\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + 1\right]$

$= 9 - \frac{9}{2} + 3 - \frac{1}{3} + \frac{1}{2} - 1$

$= \frac{54}{6} - \frac{27}{6} + \frac{18}{6} - \frac{2}{6} + \frac{3}{6} - \frac{6}{6} = \frac{40}{6} = \boxed{\frac{20}{3}}$





$A'(x) = x^2 + 1$

$A(x) = \frac{1}{3}x^3 + x$

Break it into 2 scenarios and subtract

$A'(x) = x$

$A(x) = \frac{1}{2}x^2$

$F(3) - F(1) = \frac{9}{2} - \frac{1}{2} = 4$

$A(3) = F(3) - F(1)$

$9 + 3 - [\frac{1}{3} + 1]$

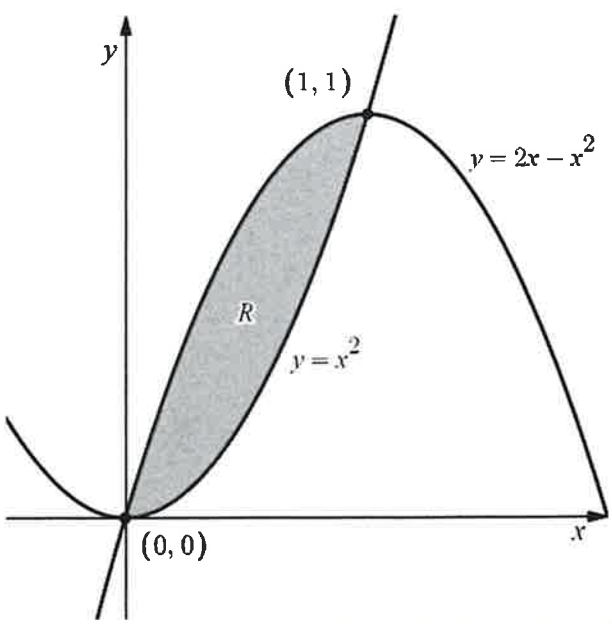
$12 - 4/3 = \boxed{32/3}$

$A = 32/3 - 4$

$= 32/3 - 12/3 = \boxed{20/3}$

Ex. 2

Find the area of the region bounded by the parabolas $y = x^2$ and $y = 2x - x^2$.



$y = x^2$ below $y = 2x - x^2$

and we need the points of intersection

$x^2 = 2x - x^2 \rightarrow 2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 0 \qquad x = 1$

$y = 0 \qquad y = 1$

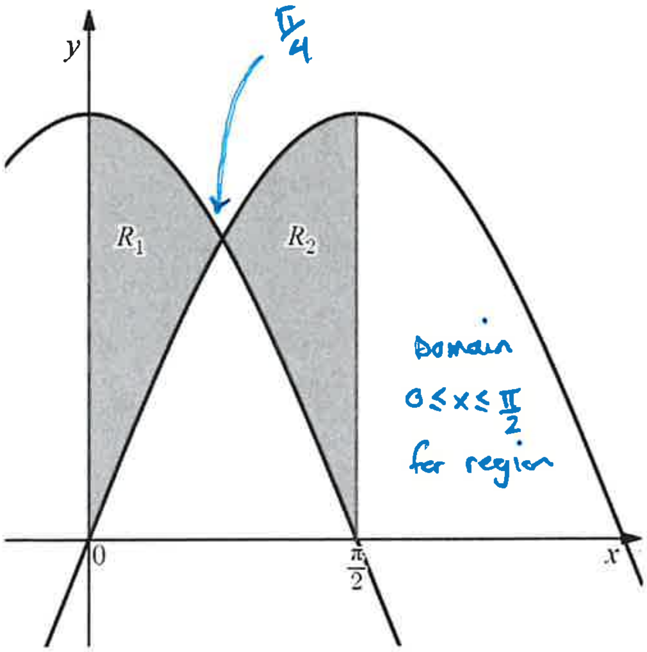
$A'(x) = 2x - x^2 - x^2$

$= 2x - 2x^2 \rightarrow A(x) = x^2 - \frac{2}{3}x^3$

$A(1) = F(1) - F(0)$

$= 1^2 - \frac{2}{3}(1)^3 - [0^2 - \frac{2}{3}(0)^3] \rightarrow 1 - 2/3 - 0 \rightarrow \boxed{\frac{1}{3}}$

Ex. 3
Find the area of the region between the curves $y = \sin x$ and $y = \cos x$ from 0 to $\pi/2$.



Points of intersection

$\sin x = \cos x$ this occurs at $\frac{\pi}{4}$

For R_1: $\cos x$ above $\sin x$

$A_1'(x) = \cos x - \sin x$

$A_1(x) = \sin x + \cos x$

R_1: $F(\frac{\pi}{4}) - F(0)$

$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} - [\sin 0 + \cos 0]$

$\rightarrow \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - [0 + 1]$

$\rightarrow \frac{2}{\sqrt{2}} - 1 \rightarrow \boxed{\sqrt{2} - 1}$

R_2: $\sin x$ above $\cos x$

$A_2'(x) = \sin x - \cos x$

$A_2(x) = -\cos x - \sin x$

From: $\frac{\pi}{2}$ to $\frac{\pi}{4}$

$F_2(\frac{\pi}{2}) - F_2(\frac{\pi}{4})$

$-\cos \frac{\pi}{2} - \sin \frac{\pi}{2} - \left[-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}\right]$

$0 - (1) + [\cos \frac{\pi}{4} + \sin \frac{\pi}{4}]$

$-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$-1 + \sqrt{2}$

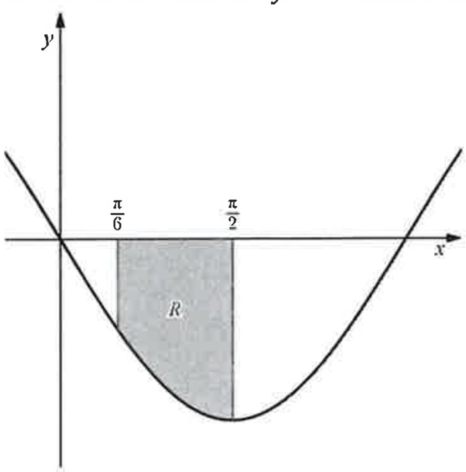
$\sqrt{2} - 1$

$R_1 + R_2 = \sqrt{2} - 1 + \sqrt{2} - 1$

$= \boxed{2\sqrt{2} - 2}$

Ex. 4

Find the area between $y = -\sin x$ and the x-axis from $\pi/6$ to $\pi/2$.



x- axis : $y=0$

$y=0$ above $y=-\sin x$

$0-(-\sin x) = \sin x$

$A'(x) = \sin x$ from $\frac{\pi}{2}$ to $\frac{\pi}{6}$

$A(x) = -\cos x$

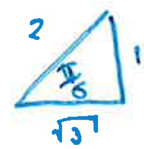
$F(\frac{\pi}{2}) - F(\frac{\pi}{6})$

$-\cos\frac{\pi}{2} - (-\cos\frac{\pi}{6})$

$-0 + \cos\frac{\pi}{6}$

$0 + \frac{\sqrt{3}}{2}$

$= \boxed{\frac{\sqrt{3}}{2}}$



Homework Assignment

- Practice Problems: #1all