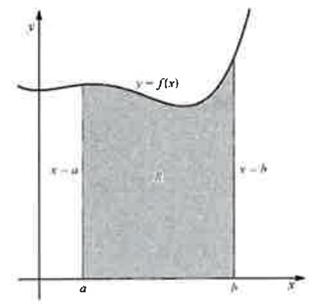
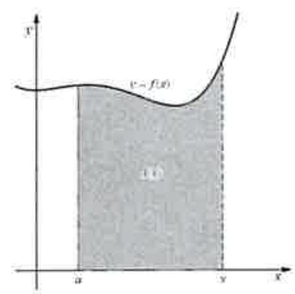
10.1 Area Under a Curve

If $y = f(x)$ is a positive function, the **area of the region under $y = f(x)$ from a to b** is the area of the region below $y = f(x)$ and above the x-axis (the line $y = 0$), to the right of the vertical line $x = a$ and to the left of $x = b$. The region is shown in the diagram to the right and defines what is called the **area problem.** Which is a problem in mathematics that existed before the tangent line problem but turns out that it is related to that problem. The issue is trying to calculate the area of a region that is curved.



An area function $A(x)$ is created to calculate the area. If a is a fixed value, then the distance x that we move to the right of a determines the area of the region.

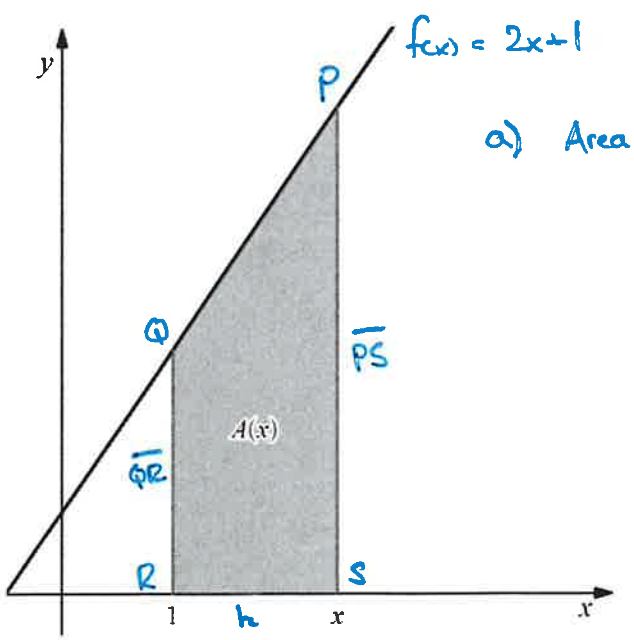


$A(x)$ is the area of the region under $y = f(x)$ from a to x. It is important to note that $A(a) = 0$.

Ex. 1

Find the area function for the region under $y = 2x + 1$ from 1 to x and compare the derivative of the area function with the equation of the straight line.

$f(1) = 3 \qquad f(x) = 2x+1$



a) Area of PQRS : $\frac{1}{2}h(\overline{PS} + \overline{QR})$

$\rightarrow \frac{1}{2}(x-1)(2x+1+3)$

$\frac{1}{2}(x-1)(2x+4)$

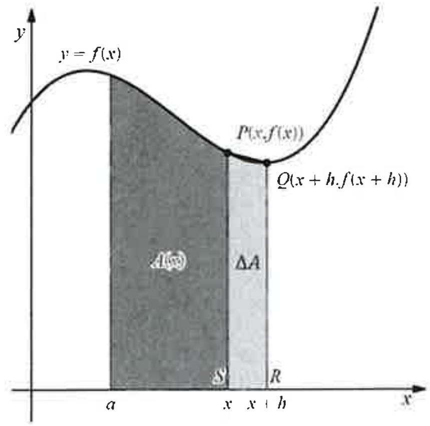
$\rightarrow (x-1)(x+2)$

$= \boxed{x^2+x-2}$

$A' = 2x+1 = f(x)$

Example 1 showed that the derivative of the area function turned out to be the function that defined the straight line. To solve the area problem, we must find an area function such that $A'(x) = f(x)$ in general.

Let $A(x)$ be the area under the function $y = f(x)$ from a to x, where $y = f(x)$ is continuous and positive.



The area can be determined by the position of the point $P\big(x, f(x)\big)$ on the curve and another point $Q\big(x+h, f(x+h)\big)$ that is very close to P. The area of the curved region $PQRS$ is the change in the area ΔA as x changes to $x + h$.

$$\begin{aligned} A'(x) &= \lim_{h\to 0}\frac{\Delta A}{h} \\ &= \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} \end{aligned}$$

The area of the curved region $PQRS$ can be approximated by the area of the trapezoid $PQRS$. As h gets smaller, the better the approximation becomes which becomes an equality in the limit as h approaches 0.

$$A(x+h) - A(x) \doteq \frac{1}{2}h[f(x) + f(x+h)]$$

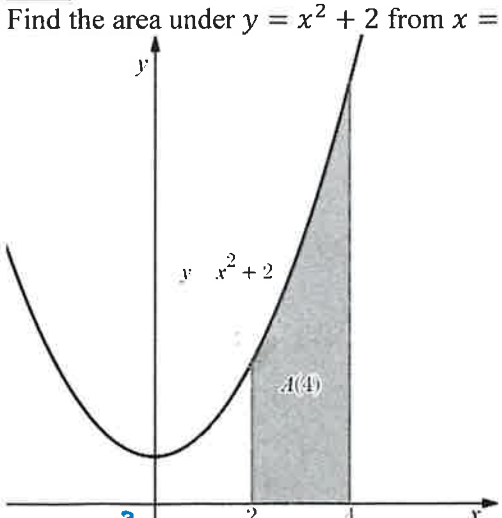
$$\begin{aligned} A'(x) &= \lim_{h\to 0}\frac{\frac{1}{2}h[f(x)+f(x+h)]}{h} \\ &= \lim_{h\to 0}\frac{f(x)+f(x+h)}{2} \\ &= \frac{f(x)+f(x)}{2} \end{aligned}$$

Therefore,

$$\boxed{A'(x) = f(x)}$$

Ex. 2

Find the area under $y = x^2 + 2$ from $x = 2$ to $x = 4$.



since $f(x) = A'(x)$

$A'(x) = x^2 + 2$ we need $A(x)$ so the antiderivative

$A(x) = \frac{1}{3}x^3 + 2x + C$ now solve for C

Initial condition is when $A(2) = 0$

$0 = \frac{1}{3}(2)^3 + 2(2) + C$

$0 = 8/3 + 4 + C$

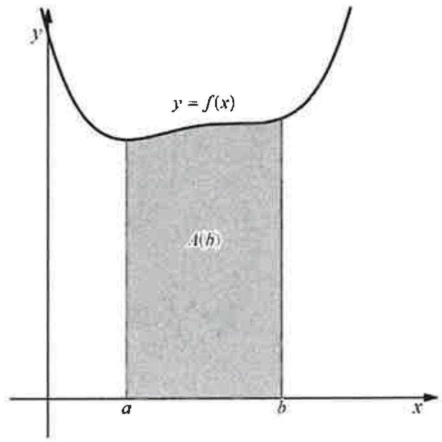
$C = -20/3$

$$\boxed{A(x) = \frac{1}{3}x^3 + 2x - \frac{20}{3}}$$

$A(4) = \frac{1}{3}(4)^3 + 2(4) - 20/3$

$= \boxed{68/3}$

Now let us reconsider the area problem. We want to find the area under $y = f(x)$ from $x = a$ to $x = b$. The required area $A(b)$ is shown in the diagram to the right. We know that the derivative of $A(x)$ is the function $f(x)$.



$$A'(x) = f(x)$$
$$A(x) = F(x) + C$$

Where $F(x)$ is the antiderivative of $f(x)$. We can solve for the value of C using the initial condition that $A(a) = 0$.

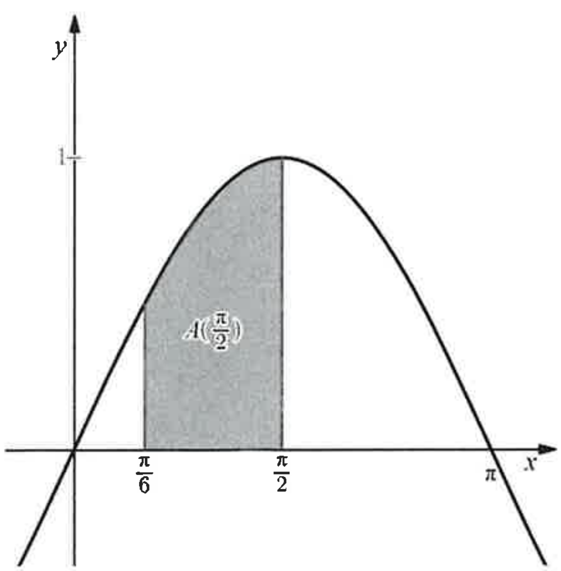
$$\begin{aligned} A(a) &= 0 \\ 0 &= F(a) + C \\ C &= -F(a) \\ A(x) &= F(x) - F(a) \\ A(b) &= F(b) - F(a) \end{aligned}$$

> If F is any antiderivative of the positive function f, the area under $y = f(x)$ from a to b is $A(b) = F(b) - F(a)$.

When using antiderivatives to calculate the area under a curve it is convenient to choose the constant C to be zero since it cancels out in the final calculation anyway. However, in other applications of antiderivatives the value of C can be found uniquely using the initial conditions of the problem. For example, if $f(x) = 3x^2 + 2$, we would choose $F(x) = x^3 + 2x$, not $F(x) = x^3 + 2x + C$ as the antiderivative to calculate the area under the curve.

Ex. 3

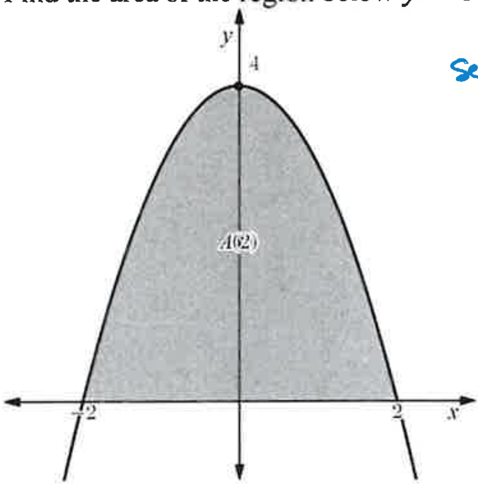
Find the area between $y = \sin x$ and the x-axis from $x = {}^{\pi}/_{6}$ to $x = {}^{\pi}/_{2}$.



$$A'(x) = f(x)$$
$$f(x) = \sin x$$
$$F(x) = -\cos x$$
$$\begin{aligned} A\left(\tfrac{\pi}{2}\right) &= F\left(\tfrac{\pi}{2}\right) - F\left(\tfrac{\pi}{6}\right) \\ &= -\cos\frac{\pi}{2} + \cos\frac{\pi}{6} \\ &= 0 + \frac{\sqrt{3}}{2} \\ &= \boxed{\frac{\sqrt{3}}{2}} \end{aligned}$$

Ex. 4

Find the area of the region below $y = 4 - x^2$ and above the x-axis.



set $y = 4 - x^2 = 0$

$0 = 4 - x^2 \qquad x = \pm 2$

helps with the given sketch.

$f(x) = 4 - x^2$

$F(x) = 4x - \frac{1}{3}x^3$

$$A(2) = F(2) - F(-2)$$

$$= 4(2) - \frac{1}{3}(2)^3 - \left[4(-2) - \frac{1}{3}(-2)^3\right]$$

$$= 8 - \frac{8}{3} - (-8 + 8/3)$$

$$= 8 - \frac{8}{3} + 8 - 8/3$$

$$= 16 - \frac{16}{3}$$

$$= \frac{48}{3} - \frac{16}{3} = \boxed{\frac{32}{3}}$$

Homework Assignment

- Practice Problems: #1abcfjk, 2, 3,4