Section 1.4 – Practice Problems

1. Determine if the Infinite Geometric Series Converges or Diverges. State the Common Ratio.

a) $16 + 4 + 1 + \cdots$

Need $-1 < r < 1$ to converge

$r = \frac{1}{4}$ Converges

b) $\frac{3}{16} + \frac{3}{8} + \frac{3}{4} + \cdots$

$r = 2$

Diverges

c) $1 + \frac{1}{1.01} + \frac{1}{(1.01)^2} + \cdots$

$r = \frac{1}{1.01}$

Converges

d) $3 - \frac{3}{2} + \frac{3}{4} - \cdots$

$r = -\frac{1}{2}$

Converges

e) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$r = \frac{1}{2}$

Converges

f) $1 + 1 + 1 + \cdots$

$r = 1$

Diverges

2. Find the sum of Infinite Geometric Series, if it exists

a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \ldots$

$r = -\frac{1}{2}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \boxed{2/3}$

b) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

$r = \frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{3}{\frac{2}{3}} = 3 \cdot \frac{3}{2} = \boxed{\frac{9}{2}}$

$= \frac{3}{1-\frac{1}{3}}$

c) $1.2 + 0.012 + 0.00012 + \cdots$

$r = \frac{1}{100}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1.2}{1-\frac{1}{100}}$

$\frac{1.2}{\frac{99}{100}}$

$= \boxed{\frac{120}{99}} = \boxed{\frac{40}{33}}$

d) $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} \ldots$

$r = \frac{3}{2}$

No sum it diverges

e) $\frac{27}{2} - 9 + 6 - 4 + \cdots$

$r = -\frac{2}{3}$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{\frac{27}{2}}{1+\frac{2}{3}}$

$\frac{\frac{27}{2}}{\frac{5}{3}}$

$\frac{27}{2} \cdot \frac{3}{5} = \boxed{\frac{81}{10}}$

f) $-6 + 3 - \frac{3}{2} + \frac{3}{4} - \cdots$

$r = -\frac{1}{2}$

$S_\infty = \frac{-6}{1-(-\frac{1}{2})}$

$= \frac{-6}{3/2}$

$-6 \cdot \frac{2}{3}$

$\boxed{-4}$

g) $\sqrt{2} - 2 + 2\sqrt{2} - 4 + \cdots$

$r = -\sqrt{2}$

No sum because $|r| \geq 1$

h) $(1.05)^{-1} + (1.05)^{-2} + (1.05)^{-3} + \cdots$

$r = \frac{1}{1.05}$

$S_\infty = \frac{a}{1-\frac{1}{1.05}} = \frac{\frac{1}{1.05}}{\frac{0.05}{1.05}}$

$\frac{1}{1.05} \cdot \frac{1.05}{0.05}$

$\frac{1}{0.05} = \boxed{20}$

3. Find the sum of each Infinite Geometric Series

a) $\sum_{i=1}^{\infty} 3\left(\frac{1}{2}\right)^{i}$

$a = 3\left(\frac{1}{2}\right)^{1}$

$= \frac{3}{2}$

$r = \frac{1}{2}$

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{3}{2}}{1-\frac{1}{2}} = \frac{3}{2}\cdot\frac{2}{1} = \boxed{3}$

b) $\sum_{k=1}^{\infty} 4\left(-\frac{1}{3}\right)^{k-1}$

$a = 4\left(-\frac{1}{3}\right)^{1-1}$

$= 4$

$r = -\frac{1}{3}$

$S_\infty = \frac{4}{1+\frac{1}{3}}$

$= \frac{4}{\frac{4}{3}} = 4\cdot\frac{3}{4} = \boxed{3}$

c) $\sum_{x=2}^{\infty} 3(4)^{1-x}$

$a = 3(4)^{1-2}$

$= 3(4)^{-1}$

$= 3/4$

$a_2 = 3\cdot 4^{1-3}$

$x=3$

$= 3\cdot 4^{-2}$

$= \frac{3}{16}$ so $r = \frac{1}{4}$

$S_\infty = \frac{a}{1-r}$

$\frac{3/4}{1-\frac{1}{4}} \rightarrow \frac{\frac{3}{4}}{3/4} = \boxed{1}$

d) $\sum_{k=2}^{\infty} 5(2)^{-k}$

$a = 5(2)^{-2}$

$= \frac{5}{4}$

$a_2 = 5\cdot 2^{-3}$

$a_2 = \frac{5}{8}$

$r = \frac{1}{2}$

$S_\infty = \frac{a}{1-r} \rightarrow \frac{\frac{5}{4}}{1-\frac{1}{2}} \rightarrow \frac{\frac{5}{4}}{\frac{1}{2}} \rightarrow \frac{5}{4}\cdot\frac{2}{1} = \boxed{\frac{5}{2}}$

e) $\sum_{i=1}^{\infty} (-1)^{i}\left(\frac{2}{3}\right)^{i+1}$

$a = (-1)^{1}\left(\frac{2}{3}\right)^{2}$

$-\frac{4}{9}$

$r = -2/3$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{-4/9}{1+2/3} = \frac{-\frac{4}{9}}{5/3} = \frac{-4}{3\cancel{9}}\cdot\frac{\cancel{3}}{5}$

$\boxed{-\frac{4}{15}}$

f) $\sum_{k=3}^{\infty} \left(\frac{4}{3}\right)^{1-k}$

since k is negative

$a = \left(\frac{4}{3}\right)^{1-3} = \left(\frac{4}{3}\right)^{-2}$

$= \left(\frac{3}{4}\right)^{2} = \frac{9}{16}$

$\left(\frac{4}{3}\right)^{-k+1} \rightarrow \left(\frac{4}{3}\right)^{-(k+1)} \rightarrow \left(\frac{3}{4}\right)^{k+1}$

$r = 3/4$

$S_\infty = \frac{9/16}{1-3/4}$

$S_\infty = \frac{\frac{9}{16}}{1/4} = \frac{9}{16}\cdot\frac{4}{1} = \boxed{\frac{9}{4}}$

$S_\infty = \frac{a}{1-r}$

g) $\sum_{n=1}^{\infty} 5 \cdot \left(-\frac{3}{5}\right)^{n-1}$

$a = 5 \quad r = -3/5$

$S_\infty = \frac{5}{1-(-3/5)} = \frac{5}{1+3/5}$

$= \frac{5}{8/5} = \boxed{\frac{25}{8}}$

h) $\sum_{n=3}^{\infty} \frac{6}{3^{n-1}}$

$\frac{6}{3^{n-1}} = 6 \cdot \frac{1}{3^{n-1}}$

$a = 6 \cdot \frac{1}{3^{3-1}} \rightarrow 6 \cdot \frac{1}{3^2} = \frac{6}{9} = \frac{2}{3}$

$S_\infty = \frac{2/3}{1-\frac{1}{3}} = \frac{2/3}{2/3} = \boxed{1}$

i) $\sum_{k=0}^{\infty} 5 \cdot \left(\frac{1}{8}\right)^{k}$

$a = 5$

$r = \frac{1}{8}$

$S_\infty = \frac{a}{1-r}$

$= \frac{5}{1-\frac{1}{8}} = \frac{5}{7/8}$

$= \boxed{\frac{40}{7}}$

j) $\sum_{k=3}^{\infty} 10 \cdot \left(-\frac{2}{5}\right)^{k-1}$

$a = 10 \cdot \left(-\frac{2}{5}\right)^{3-1}$

$10 \cdot \left(-\frac{2}{5}\right)^2 = 10 \cdot \frac{4}{25}$

$= \frac{40}{25} = 8/5$

$a = 8/5 \quad r = -2/5$

$S_\infty = \frac{a}{1-r} = \frac{8/5}{1-(-2/5)} = \frac{\frac{8}{5}}{7/5} = \boxed{\frac{8}{7}}$

4. Find the rational number represented by the repeating decimal.

a) $0.\overline{38}$

0.383838

$\frac{38}{100} + \frac{38}{10000} + \ldots$

$r = \frac{1}{100}$

$a = \frac{38}{100}$

$S_\infty = \frac{\frac{38}{100}}{1-\frac{1}{100}}$

$= \boxed{\frac{38}{99}}$

b) $0.3\overline{8}$

$0.3888\ldots$

$\frac{3}{10} + \underbrace{\frac{8}{100} + \frac{8}{1000} + \ldots}_{\text{starts here}}$

$a = \frac{8}{100} \quad r = \frac{1}{10}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{8}{100}}{1-\frac{1}{10}} = 4/45$

$\frac{3}{10} + \frac{4}{45}$

$\downarrow$

$\boxed{\frac{7}{18}}$

c) $1.\overline{432}$

$1.432432432\ldots$

$1 + \frac{432}{1000} + \frac{432}{1000000} + \ldots$

$r = \frac{1}{1000}$

$a = \frac{432}{1000}$

$S_\infty = \frac{\frac{432}{1000}}{1 - \frac{1}{1000}} = \frac{16}{37}$

$1 + \frac{16}{37} = \boxed{\frac{53}{37}}$

d) $1.4\overline{32}$

1.4323232

$1 + \frac{4}{10} + \underbrace{\frac{32}{1000} + \frac{32}{100000} + \ldots}_{\text{repeats}}$

$a = \frac{32}{1000}$

$r = \frac{1}{100}$

$S_\infty = \frac{\frac{32}{1000}}{1 - \frac{1}{100}} = \frac{16}{495}$

$1 + \frac{4}{10} + \frac{16}{495} = \boxed{\frac{709}{495}}$

5. Solve for x.

a) $\sum_{k=1}^{\infty} x^k = \frac{2}{5}$

$x + x^2 + x^3 + \ldots$

$S_\infty = 2/5$

$r = x$

$a = x$

$\frac{2}{5} = \frac{x}{1-x} \rightarrow 2(1-x) = 5x$

$2 - 2x = 5x$

$2 = 7x$

$\boxed{x = 2/7}$

b) $\sum_{i=1}^{\infty} (2x)^{i-1} = \frac{2}{5}$

$1 + 2x + 4x^2 + 8x^3 \ldots$

$a = 1$

$r = 2x$

$S_\infty = \frac{1}{1-2x} \rightarrow \frac{2}{5} = \frac{1}{1-2x}$

$2(1-2x) = 5$

$2 - 4x = 5$

$-4x = 3$

$\boxed{x = -3/4}$

$\left[2(-3/4) = -\frac{3}{2}\right]$

$r < -1$ so $\emptyset$

c) $\sum_{j=0}^{\infty} (\cos x)^j = 2, \quad 0° \le x < 360°$

$1 + \cos x + (\cos x)^2 + \ldots$

$r = \cos x$

$S_\infty = \frac{a}{1-r} \rightarrow 2 = \frac{1}{1 - \cos x}$

$2(1 - \cos x) = 1$

$2 - 2\cos x = 1$

$2 - 1 = 2\cos x$

$\frac{1}{2} = \cos x$

by special angles Q4, Q1

$x = 60°$ or $300°$

In Q1 $x = 60°$ Q4 $x = 300°$

d) $\sum_{n=1}^{\infty} 15x(x^2)^{n-1} = 4$

$15x + 15x^3 + 15x^5 + \ldots$

$a = 15x$

$r = x^2$

$4 = \frac{15x}{1-x^2} \rightarrow 4(1-x^2) = 15x$

$4 - 4x^2 = 15x \rightarrow 4x^2 + 15x - 4 = 0$

$4x^2 + 16x - x - 4 = 0$

$4x(x+4) - 1(x+4) = 0$

$(4x-1)(x+4) = 0$

$x = -4$

$x = \frac{1}{4}$ ← only this one.

when $x = -4$

$r = 16$ so reject

6. Rewrite each series using the new index.

a) $\sum_{n=2}^{\infty} 5^n \cdot 2^{-n} = \sum_{j=5}^{\infty} 5^{j-3} \cdot 2^{-j+3}$

$j-3=2 \quad -j+3=-2$

$5^2 \cdot 2^{-2}$

b) $\sum_{i=0}^{\infty} 2^{3i-1} = \sum_{j=4}^{\infty} 2^{3j-13}$

$2^{3(0)-1}$ $\quad 3j-13=-1$

2^{-1}

c) $\sum_{k=1}^{\infty} 3^{1-k} = \sum_{k=3}^{\infty} 3^{3-k}$

3^{1-1}

3^0

1

d) $\sum_{x=5}^{\infty} 4^{2-3x} = \sum_{x=1}^{\infty} 4^{-10-3x}$

$4^{2-3(5)}$

4^{2-15}

4^{-13}

$y-3x=-13$

when $x=1$

$y=-10$

7. For what value will the infinite Geometric Series, $1+(1+x)+(1+x)^2+\cdots, x \neq 1$, have a finite sum?

need $-1<r<1$

$r=1+x$

so $-1<1+x<1$

$\boxed{-2<x<0}$

8. Determine the value of $x, x \neq 0$, so that the infinite Geometric Series, $1+\frac{1}{3}x+\frac{1}{9}x^2+\cdots$, has a finite sum

$r=\frac{1}{3}x$

$-1<r<1$

$-1<\frac{1}{3}x<1$

$\boxed{-3<x<3}$

9. Why is it impossible to have an Infinite Geometric Series with first term 9 and a sum of 4?

$S_\infty=4$

$a=9$

$S_\infty=\frac{a}{1-r}$

$4=\frac{9}{1-r}$

$4(1-r)=9$

$4-4r=9$

$-4r=5$

$r=-5/4$ ← outside of convergence Domain, need $-1<r<1$

10. The first term in an infinite Geometric Series is 3. Find all the possible values of the common ratio r, which will give a sum greater than 4.

$4<\frac{a}{1-r} \rightarrow 4<\frac{3}{1-r}$

$4(1-r)<3$

$4-4r<3$

$4-3<4r$

$1<4r$

$\boxed{\frac{1}{4}<r \text{ but still } <1}$

 * Now $S_\infty = \frac{2}{1-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = \boxed{\frac{5}{2}}$

11. Find the sum of an Infinite Geometric Series if the first term is 2, and each term is 4 times the sum of all the terms that follow it (Challenging).

$a = 2 = 4(ar + ar^2 + ar^3 + \dots)$

since $a = 2$ we get

$a = 4(2r + 2r^2 + 2r^3 + \dots)$

$2 = 8(r + r^2 + r^3 + \dots)$

$\frac{2}{8} = r + r^2 + r^3 + \dots) \Rightarrow \frac{1}{4} = r + r^2 + \dots$

For the sequence then $a = r$

$\frac{1}{4} = \frac{r}{1-r}$

$\frac{1}{4} - \frac{1}{4}r = r$

$\frac{1}{4} = r + \frac{1}{4}r$

$\frac{1}{4} = \frac{5}{4}r$

$\frac{4}{5} \cdot \frac{1}{4} = r$

$\boxed{r = \frac{1}{5}}$ *

12. A weather balloon rises $100m$ the first minute, and each minute after the first it rises 4% les than the previous minute. What is the maximum height reached by the balloon.

$a = 100$

$r = 96\%$ or 0.96

$S_\infty = \frac{a}{1-r} = \frac{100}{1-0.96} = \frac{100}{0.04}$

$= \boxed{2500m}$

13. If a rectangle has dimensions 2 x 4, and each side is halved endlessly, what is the total area of all the rectangles formed if the process is continued without end?

$A = l \cdot w$

$A = 8$

if we halve both sides area reduces by factor of $\frac{1}{4}$

$a = 8$

$r = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$S_\infty = \frac{8}{1-\frac{1}{4}} = \frac{8}{\frac{3}{4}} = 8 \cdot \frac{4}{3}$

$= \boxed{\frac{32}{3}}$

14. A $6.25cm$ nail is driven $2cm$ into a board on the first hit, and $\frac{2}{3}$ of the remaining distance on each of the next hits. How many hits are required to hammer the nail into the board.

$a = 6.25 - 2$

$r = \frac{2}{3}$

$S_\infty = \frac{4.25}{1-\frac{2}{3}}$

$= \frac{4.25}{\frac{1}{3}}$

$= 12.75 \rightarrow$ 13 full hits

by this formula we would never completely hit the nail into the board fully.

But in reality of course we would.

15. A clock pendulum swings through an arc of $30cm$. Each successive swing is 10% less. How far will the pendulum swing altogether before coming to a complete stop?

$a = 30$

$r = 90\%$ or 0.90

$S_\infty = \frac{30}{1-0.90} = \frac{30}{0.10} = \boxed{300cm}$

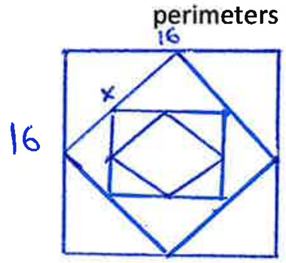
16. Seventy-two grams of fertilizer is given to a dogwood tree the first year, 60 *grams* the second year, 50 *grams* the third year, and so on. Find the total amount of fertilizer received by the dogwood tree?

$72, 60, 50, \ldots \quad r = 5/6 \quad a = 72$

$S_\infty = \frac{72}{1-5/6} = \frac{72}{1/6} = 72 \cdot 6$

$= \boxed{432g}$

17. The side of a square is $16cm$. The midpoint of the sides of the square are joined to form another inscribed square. This process is continued forever. Find the sum of the perimeters of all the squares.



and so on

$P_1 = 64$

$P_2 = 8^2 + 8^2 = x^2$ (Pythagorean Theorem)

$= 128 = x^2$

$x = \sqrt{128}$

$= 8\sqrt{2}$

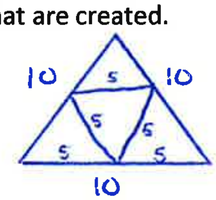
$P_2 = 4(8\sqrt{2}) = 32\sqrt{2}$

$r = \frac{32\sqrt{2}}{64} = \frac{\sqrt{2}}{2}$

$S_\infty = \frac{64}{1-\frac{\sqrt{2}}{2}}$

$= \boxed{218.51cm}$

18. A side of an equilateral triangle is $10cm$. The midpoints of the side are joined to form a second inscribed equilateral triangle inside the first one, and this process is continued infinitely. Find the sum of the perimeter of the original triangle and all the additional triangles that are created.



$P_1 = 30$

$P_2 = 15$

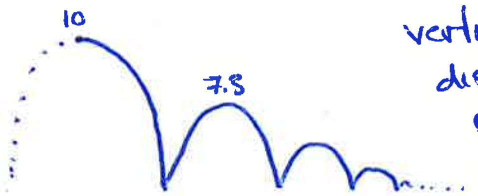
$r = \frac{1}{2}$

$S_\infty = \frac{30}{1-\frac{1}{2}} = \frac{30}{\frac{1}{2}}$

$= \boxed{60cm}$

19. A ball dropped from $10m$ above the ground rebounds 75% of the distance of the height it fell from.

a) Determine the total vertical distance traveled when the ball comes to rest.



vertical distance consider full up/down

$a = 20$

$r = 0.75$

$$S_\infty = \frac{20}{1-0.75} - 10$$

the non existed initial up

$$= 80 - 10 = \boxed{70m}$$

b) Determine the total vertical distance traveled by the ball after 10 $bounces$.

$$S_{10} = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{20(1-0.75^{10})}{1-0.75} - 10$$

$$= 75.5 - 10 = \boxed{65.5m}$$

c) What is the distance travelled after the 10th bounce.

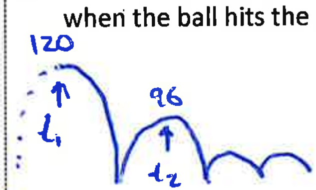
$$S_\infty - S_{10}$$

$$70 - 65.5$$

$$= \boxed{4.5m}$$

20. A ball dropped from a height of $120m$ rebounds 80% of the distance of the height it previously fell.

a) Determine the total vertical distance traveled when the ball hits the ground the 8th time.



again start from ground

$a = 240$

$r = 0.80$

$$S_8 = \frac{240(1-0.80^8)}{1-0.80} - 120$$

$$= 998.67 - 120$$

$$= \boxed{878.67m}$$

b) Determine the maximum height reached after hitting the ground the 8th time.

$$t_8 = ar^{n-1}$$

start at 120 this time

$$= 120(0.8)^{8-1}$$

$$= 120(0.8)^7$$

$$= \boxed{25.2}$$

c) Determine the total vertical distance that the ball travels before coming to rest.

$$S_\infty = \frac{240}{1-0.80} - 120$$

$$= 1200 - 120$$

$$= \boxed{1080m}$$

See Website for the Detailed Answer Key