Section 1.4 – Infinite Geometric Series

Consider standing $1ft$ away from a wall and always moving half the distance to the wall.

- It can be represented by this Geometric Series:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{32}, \dots, \left(\frac{1}{2}\right)^{n-1}$$

- Mathematically, we would never reach the wall, never get to zero, progress infinitely.
- But as terms of an infinite Geometric Series get closer and closer to a real number, in this case 0. The sequence is said to **Converge** at that real number, and we call it **Convergent**.

Consider what happens as n gets larger and larger.

$$\left(\frac{1}{2}\right)^{50} = 8.8 \cdot 10^{-16}, \quad \left(\frac{1}{2}\right)^{100} = 7.88 \cdot 10^{-31}, \quad \left(\frac{1}{2}\right)^{200} = 6.22 \cdot 10^{-61}$$

As n gets infinitely large, $\left(\frac{1}{2}\right)^{n}$, gets closer and closer to 0.

Here we see the first hints of Calculus, we say that:

$$\lim_{n \to \infty} \left(\frac{1}{2}\right)^{n} = 0$$

- As $\boldsymbol{n}$ **approaches** an infinitely large number, $\left(\frac{1}{2}\right)^{n}$ approaches zero**.**

Now consider the infinite Geometric Sequence: $1, 2, 4, 8, 16, \dots, 2^{n-1}$

- This sequence does not converge to some number, it gets infinitely large.
- A sequence like this **Diverges** and is said to be **Divergent**

So what is the difference?

- It all comes down to the $r - value$
  - Infinite Geometric Sequences that have $r - values$ between $-1 < r < 1$ will **Converge**
  - Infinite Geometric Sequences that have $r - values$ where $r > 1$ or $r < -1$ will **Diverge**

The Sum of an Infinite Geometric Sequence

The sum of the terms of an infinite Geometric Sequence is given by:

$$S_n = \frac{a}{1-r}, \ for -1 < r < 1 \quad or \quad |r| < 1$$

Example 1: Find the sum of the infinite Geometric Series $\frac{2}{5} - \frac{4}{25} + \frac{8}{125} - \cdots$

Solution 1: First determine r

$$r = \frac{\frac{-4}{25}}{\frac{2}{5}} = -\frac{2}{5} \quad and\ is -1 < -\frac{2}{5} < 1, so\ it\ is\ a\ Finite\ Sum$$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{2}{5}}{1-\left(-\frac{2}{5}\right)} = \frac{\frac{2}{5}}{\frac{7}{5}} = \frac{2}{7}$$

Sigma Notation of an Infinite Geometric Series

An infinite Geometric Series with first term a and a common ratio r, $|r| < 1$, is denoted by:

$$\sum_{i=1}^{\infty} ar^{i-1}$$

Example 2: Find the sum of $\sum_{k=2}^{\infty} 8\left(-\frac{1}{2}\right)^{k-1}$

Solution 2: First find a, it is the first term, so set $k = 2$ in this case

$$a = 8\left(-\frac{1}{2}\right)^{2-1} = -4 \qquad and \qquad r = -\frac{1}{2}$$

$$S_\infty = \frac{a}{1-r} = \frac{-4}{1-\left(-\frac{1}{2}\right)} = \frac{-4}{\frac{3}{2}} = -\frac{8}{3}$$

Example 3: A pendulum swings through an arc of $25cm$. On each successive swing friction lessens the length of the swing by 5%. When the pendulum stops, what total length will the pendulum have swung?

Solution 3: If the length of the swing is lessened by 5%, then the swing is 95% its original arc.

$$S_\infty = \frac{a}{1-r} = \frac{25}{1-0.95} = \frac{25}{0.05} = 500cm$$

Example 4: A ball is dropped from $12ft$ and rebounds two-thirds the distance from which it fell. Find the total distance the ball travelled.

Solution 4: Remember that we have to consider the down/up motion, so we will have to **subtract the initial up value of $\boldsymbol{12ft}$.**

$$a = 24ft \left(includes\ the\ \frac{up}{down}\right) \qquad r = \frac{2}{3}$$

$$S_\infty = \frac{a}{1-r} - \mathbf{12} = \frac{24}{1-\frac{2}{3}} - \mathbf{12} = \frac{24}{\frac{1}{3}} = 60ft$$

Example 5: Write the repeating decimal $0.2\overline{4}$ as a fraction

Solution 5: In this case consider, $0.2\overline{4}\ is\ 0.2444444\ \ldots$ we need to consider place holders

$$0.2\overline{4} = \frac{2}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10\,000} + \cdots$$

This is the repeating part

$$r = \frac{1}{10}$$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{2}{10} + \frac{\frac{4}{100}}{1-\frac{1}{10}}$$

$$= \frac{2}{10} + \frac{\frac{1}{25}}{\frac{9}{10}}$$

$$= \frac{2}{10} + \frac{1}{25} \cdot \frac{10}{9}$$

$$= \frac{11}{45}$$

Section 1.4 – Practice Problems

1. Determine if the Infinite Geometric Series Converges or Diverges. State the Common Ratio.

a) $16 + 4 + 1 + \cdots$

b) $\frac{3}{16} + \frac{3}{8} + \frac{3}{4} + \cdots$

c) $1 + \frac{1}{1.01} + \frac{1}{(1.01)^2} + \cdots$

d) $3 - \frac{3}{2} + \frac{3}{4} - \cdots$

e) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

f) $1 + 1 + 1 + \cdots$

2. Find the sum of Infinite Geometric Series, if it exists

a) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\ldots$

b) $3+1+\frac{1}{3}+\frac{1}{9}+\cdots$

c) $1.2+0.012+0.00012+\cdots$

d) $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}\ldots$

e) $\frac{27}{2}-9+6-4+\cdots$

f) $-6+3-\frac{3}{2}+\frac{3}{4}-\cdots$

g) $\sqrt{2}-2+2\sqrt{2}-4+\cdots$

h) $(1.05)^{-1}+(1.05)^{-2}+(1.05)^{-3}+\cdots$

3. Find the sum of each Infinite Geometric Series

a) $\sum_{i=1}^{\infty} 3\left(\frac{1}{2}\right)^{i}$

b) $\sum_{k=1}^{\infty} 4\left(-\frac{1}{3}\right)^{k-1}$

c) $\sum_{x=2}^{\infty} 3(4)^{1-x}$

d) $\sum_{k=2}^{\infty} 5(2)^{-k}$

e) $\sum_{i=1}^{\infty} (-1)^{i}\left(\frac{2}{3}\right)^{i+1}$

f) $\sum_{k=3}^{\infty} \left(\frac{4}{3}\right)^{1-k}$

g) $\sum_{n=1}^{\infty} 5 \cdot \left(-\frac{3}{5}\right)^{n-1}$

h) $\sum_{n=3}^{\infty} \frac{6}{3^{n-1}}$

i) $\sum_{k=0}^{\infty} 5 \cdot \left(\frac{1}{8}\right)^{k}$

j) $\sum_{k=3}^{\infty} 10 \cdot \left(-\frac{2}{5}\right)^{k-1}$

4. Find the rational number represented by the repeating decimal.
   a) $0.\overline{38}$
   b) $0.3\overline{8}$

c) $1.\overline{432}$

d) $1.4\overline{32}$

5. Solve for x.

a) $\sum_{k=1}^{\infty} x^k = \frac{2}{5}$

b) $\sum_{i=1}^{\infty} (2x)^{i-1} = \frac{2}{5}$

c) $\sum_{j=0}^{\infty} (\cos x)^j = 2, \quad 0° \le x < 360°$

d) $\sum_{n=1}^{\infty} 15x(x^2)^{n-1} = 4$

6. Rewrite each series using the new index.

a) $\sum_{n=2}^{\infty} 5^n \cdot 2^{-n} = \sum_{j=5}^{\infty}$

b) $\sum_{i=0}^{\infty} 2^{3i-1} = \sum_{j=4}^{\infty}$

c) $\sum_{k=1}^{\infty} 3^{1-k} = \sum_{k=3}^{\infty}$

d) $\sum_{x=5}^{\infty} 4^{2-3x} = \sum_{x=1}^{\infty}$

7. For what value will the infinite Geometric Series, $1 + (1 + x) + (1 + x)^2 + \cdots, x \neq 1$, have a finite sum?

8. Determine the value of $x, x \neq 0$, so that the infinite Geometric Series, $1 + \frac{1}{3}x + \frac{1}{9}x^2 + \cdots$, has a finite sum

9. Why is it impossible to have an Infinite Geometric Series with first term 9 and a sum of 4?

10. The first term in an infinite Geometric Series is 3. Find all the possible values of the common ratio r, which will give a sum greater than 4.

11. Find the sum of an Infinite Geometric Series if the first term is 2, and each term is 4 times the sum of all the terms that follow it (Challenging).

12. A weather balloon rises $100m$ the first minute, and each minute after the first it rises 4% les than the previous minute. What is the maximum height reached by the balloon.

13. If a rectangle has dimensions $2\ x\ 4$, and each side is halved endlessly, what is the total area of all the rectangles formed if the process is continued without end?

14. A $6.25cm$ nail is driven $2cm$ into a board on the first hit, and $\frac{2}{3}$ of the remaining distance on each of the next hits. How many hits are required to hammer the nail into the board.

15. A clock pendulum swings through an arc of $30cm$. Each successive swing is 10% less. How far will the pendulum swing altogether before coming to a complete stop?

16. Seventy-two grams of fertilizer is given to a dogwood tree the first year, $60\ grams$ the second year, $50\ grams$ the third year, and so on. Find the total amount of fertilizer received by the dogwood tree?

17. The side of a square is $16cm$. The midpoint of the sides of the square are joined to form another inscribed square. This process is continued forever. Find the sum of the perimeters of all the squares.

18. A side of an equilateral triangle is $10cm$. The midpoints of the side are joined to form a second inscribed equilateral triangle inside the first one, and this process is continued infinitely. Find the sum of the perimeter of the original triangle and all the additional triangles that are created.

19. A ball dropped from $10m$ above the ground rebounds 75% of the distance of the height it fell from.

a) Determine the total vertical distance traveled when the ball comes to rest.

b) Determine the total vertical distance traveled by the ball after $10\ bounces$.

c) What is the distance travelled after the 10^{th} bounce.

20. A ball dropped from a height of $120m$ rebounds 80% of the distance of the height it previously fell.

a) Determine the total vertical distance traveled when the ball hits the ground the 8^{th} time.

b) Determine the maximum height reached after hitting the ground the 8^{th} time.

c) Determine the total vertical distance that the ball travels before coming to rest.

See Website for the Detailed Answer Key

Extra Work Space