$S_n = \frac{a(1-r^n)}{1-r}$ or $S_n = \frac{a-rl}{1-r}$

Section 1.3 – Practice Problems

1. Determine the desired information.

a) $S_{10},\ if\ a = 8, r = \frac{1}{2}$

$S_{10} = \frac{8(1-\frac{1}{2}^{10})}{1-\frac{1}{2}}$

$\boxed{S_{10} = 15.98}$

b) $S_9,\ if\ a = -6, r = 2$

$S_9 = \frac{-6(1-2^9)}{1-2}$

$\boxed{S_9 = -3066}$

c) $a, if\ S_8 = 765, r = 2$

$S_8 = \frac{a(1-2^8)}{1-2}$

$765 = \frac{a(-255)}{-1}$

$765 = 255a$

$\boxed{a = 3}$

d) $S_6,\ if\ a = -8,\ t_4 = 27$

$S_6 = \frac{a(1-r^n)}{1-r}$

need r

$t_n = ar^{n-1}$

$27 = -8(r)^3$

$-\frac{27}{8} = r^3$

$r = \sqrt[3]{-\frac{27}{8}} = -\frac{3}{2}$

$S_6 = \frac{-8(1-(-\frac{3}{2})^6)}{1-(-3/2)}$

$\boxed{S_6 = 33.25}$

e) S_5, if $t_3 = 3, r = \frac{1}{2}$

need a

$ar^2 = 3$

$a\left(\frac{1}{2}\right)^2 = 3$

$a\left(\frac{1}{4}\right) = 3$

$a = 12$

$S_5 = \frac{a(1-(\frac{1}{2})^5)}{1-\frac{1}{2}} = \frac{12(1-(\frac{1}{2})^5)}{1-\frac{1}{2}}$

$\boxed{S_5 = 23.25}$

f) S_8, if $a = 12, t_5 = 192$

$ar^4 = 192 \rightarrow 12r^4 = 192$

$r^4 = 16$

$r = \pm 2$

if $r = 2$ $\quad S_8 = \frac{12(1-2^8)}{1-2} = \boxed{3060}$

if $r = -2$ $\quad S_8 = \frac{12(1-(-2)^8)}{1-(-2)} = \boxed{-1020}$

g) t_3, if $S_5 = 93, r = 2$

$S_5 = \frac{a(1-2^5)}{1-2}$

$93 = \frac{a(1-32)}{-1} \rightarrow 93 = \frac{a(-31)}{-1}$

$93 = 31a$

$a = 3$

$a \cdot r^2 = t_3$

$3 \cdot 2^2 = t_3$

$\boxed{12 = t_3}$

h) r, if $a = 3, S_3 = 39$

$S_3 = \frac{3(1-r^3)}{1-r}$

$39 = \frac{3-3r^3}{1-r} \rightarrow 39(1-r) = 3-3r^3$

$39 - 39r = 3 - 3r^3$

$3r^3 - 39r = 3 - 39$

$3r^3 - 39r = -36$

Divide everything by 3

$r^3 - 13r = -12$

$r(r^2-13) = -12$

Try $3 \cdot 4$

$2 \cdot 6$

~~1 12~~ not allowed

$\boxed{r = 3 \text{ or } r = -4}$

is the only combo that works

$r \neq 1$ Domain restriction

$S_n = \frac{a(1-r^n)}{1-r}$ or $S_n = \frac{a-rl}{1-r}$

i) $S_{100},\ if\ t_1=-1, t_2=1, t_3=-1, t_4=1$

$r=-1$
$a=-1$
$n=100$

$S_{100} = \frac{a(1-r^n)}{1-r}$

$S_{100} = \frac{-1(1-(-1)^{100})}{1-(-1)}$

$= \frac{-1(0)}{2}$

$= \boxed{0}$

j) $S_{101},\ if\ t_1=-1, t_2=1, t_3=-1, t_4=1$

Same pattern as here

just added one more odd # term

so $\boxed{-1}$

2. Find the number of terms using the information given.

a) $\sum_{k=8}^{35}(2^k-5)$

$n = 35-8+1$

$\boxed{n=28}$

b) $\sum_{i=8}^{b}(2^i-5)$

$b-8+1=n$

$\boxed{b-7=n}$

c) $\sum_{k=a}^{9}(2^k-5)$

$9-a+1=n$

$\boxed{10-a=n}$

d) $\sum_{i=a}^{b}(2^i-5)$

$\boxed{b-a+1=n}$

$S_n = \frac{a(1-r^n)}{1-r}$ or $S_n = \frac{a - r\ell}{1-r}$

3. Find the sum of each Geometric Series

a. $\sum_{k=1}^{8} 3 \cdot (2)^{k-1}$

when $k=1$ $a=3$

$r=2$

$n = 8-1+1$
$= 8$

$S_8 = \frac{3(1-2^8)}{1-2}$

$\boxed{S_8 = 765}$

b. $\sum_{k=1}^{12} 2 \cdot (-3)^{k-1}$

when $k=1$ $a=2$

when $k=12$ $a_{12} = -354294$ (this is ℓ)

$S_{12} = \frac{2-(-3)(-354294)}{1-(-3)}$

$= \boxed{-265720}$

c. $\sum_{x=0}^{10} 5 \cdot \left(\frac{1}{2}\right)^x$

$n = 11$

r

when $x=0$ $a=5$

$S_{11} = \frac{5(1-(\frac{1}{2})^{11})}{1-(\frac{1}{2})} = \boxed{9.9995}$

$\boxed{\text{10 is fine}}$

d. $\sum_{k=0}^{9} \frac{3}{5^{k+1}}$

when $k=0$ $a = 3/5$ $k=1$ $a_2 = \frac{3}{25}$

$n=10$

$r = \frac{1}{5}$

$S_{10} = \frac{\frac{3}{5}(1-(\frac{1}{5})^{10})}{1-\frac{1}{5}}$

$\boxed{S_{10} = 0.75}$

e. $\sum_{b=2}^{9} 18 \cdot (0.1)^b$

$n=8$ when $b=2$ $18 \cdot 0.1^2$

$a_1 = 0.18$

$S_n = \frac{a(1-r^n)}{1-r}$ $b=3$ $18 \cdot 0.1^3$

$a_2 = 0.018$

$S_8 = \frac{0.18(1-(0.1)^8)}{1-0.1}$ $r = \frac{0.018}{0.18} = 0.1$

$= \boxed{0.2}$

f. $\sum_{i=3}^{11} \frac{2^i}{3^{i-1}}$

$n=9$ $i=3 \rightarrow \frac{2^3}{3^{3-1}} = \frac{8}{9} = \frac{8}{9}$

$r = 2/3$

$a = \frac{8}{9}$ $i=4 \rightarrow \frac{2^4}{3^{4-1}} = \frac{16}{27} = \frac{16}{27}$

$S_9 = \frac{\frac{8}{9}(1-(\frac{2}{3})^9)}{1-\frac{2}{3}}$ $r = \frac{16}{27} \div \frac{8}{9}$

$\boxed{S_9 = 2.597}$ $r = \frac{16}{27} \cdot \frac{9}{8} = \boxed{\frac{2}{3}}$

$\boxed{\text{2.6 is fine}}$

$S_n = \frac{a(1-r^n)}{1-r}$ or $S_n = \frac{a - r\ell}{1-r}$

$3 \cdot 3^{n-1}$
$= 3 \cdot 3^n \cdot 3^{-1}$
$= 3^n$

1. $\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k$

$n = n-1+1$
$n = n$
$r = \frac{2}{3}$

when $k=1$ $a_1 = \frac{2}{3}$
$k=2$ $a_2 = \frac{4}{9}$

$S_n = \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} = \frac{2\left(1-\left(\frac{2}{3}\right)^n\right)}{3\left(\frac{1}{3}\right)}$

$\boxed{2\left(1-\left(\frac{2}{3}\right)^n\right)}$

1. $\sum_{k=1}^{n} 4 \cdot 3^{k-1}$

$n = n$
$r = 3$
$a = 4$

$k=1$ $a_1 = 4$
$k=2$ $a_2 = 12$
$k=n$ $a_n = \ell = 4 \cdot 3^{n-1}$

$S_n = \frac{4 - 3(4 \cdot 3^{n-1})}{1-3}$

$S_n = \frac{4 - 4 \cdot 3^n}{-2}$

$S_n = \frac{4(1-3^n)}{-2}$

$-2(1-3^n)$
$= -2(-3^n+1)$
$= \boxed{2(3^n-1)}$

4. Write the Geometric Series using Sigma Notation with $k = 1$ as the index of the first term.

a) $3 + 6 + 12 + 24 + 48$

$a = 3$ $r = 2$
$n = 5$

$\boxed{\sum_{k=1}^{5} 3 \cdot 2^{k-1}}$

b) $2 - 6 + 18 - 54 + \cdots + 13\,122$

$a = 2$
$n = 9$
$r = -3$

$\boxed{\sum_{k=1}^{9} 2 \cdot (-3)^{k-1}}$

Find n
$t_n = ar^{n-1}$
$13122 = 2(-3)^{n-1}$
$6561 = (-3)^{n-1}$
$3^8 = (-3)^{n-1}$
$n = 9$
just tried powers of 3

c) $\frac{3}{16} - \frac{3}{8} + \frac{3}{4} - \cdots - 384$

$a = \frac{3}{16}$ $n = 12$
$r = -2$

$\boxed{\sum_{k=1}^{12} \frac{3}{16} \cdot (-2)^{k-1}}$

Find n
$t_n = ar^{n-1}$
$-384 = \frac{3}{16} r^{n-1}$
$-2048 = (-2)^{n-1}$
$-2^{11} = (-2)^{n-1}$
$n = 12$

d) $8 + 4 + 2 + 1 + \cdots + \frac{1}{1024}$

$a = 8$ $n = 14$
$r = \frac{1}{2}$

$\boxed{\sum_{k=1}^{14} 8 \cdot \left(\frac{1}{2}\right)^{k-1}}$

$t_n = ar^{n-1}$
$\frac{1}{1024} = 8\left(\frac{1}{2}\right)^{n-1}$
$\frac{1}{8192} = \left(\frac{1}{2}\right)^{n-1}$
$\frac{1}{2^{13}} = \frac{1}{2^{n-1}}$
$n = 14$

5. Solve for a: $\sum_{i=0}^{2} a^i = 31$

$a^0 + a^1 + a^2 = 31$

$\downarrow$

$1 + a^1 + a^2 = 31$

$a^1 + a^2 = 30$

$a^2 + a - 30 = 0$

$(a+6)(a-5) = 0$

$\boxed{a = -6 \quad \text{or} \quad a = 5}$

6. Which is larger, and by what amount? $2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$, *or* 2^n

$S_{n-1} = \dfrac{a - rl}{1-r}$ $\quad a = 1 \quad l = 2^{n-1} \quad r = 2$

$S_{n-1} = \dfrac{1 - 2(2^{n-1})}{1-2} = \dfrac{1-2^n}{-1} = 2^n - 1$

so $\boxed{2^n \text{ is bigger by } 1}$

7. If 64 students enter a singles tennis tournament, where the winner of each match advances to the next round, how many matches must be played before a winner is determined?

64 pair off instadly

$32 + 16 + 8 + 4 + 2 + 1$

$a = 32$

$r = \frac{1}{2}$

$l = 1$

$S_6 = \dfrac{32 - \frac{1}{2}(1)}{1 - \frac{1}{2}} = \dfrac{32 - \frac{1}{2}}{+\frac{1}{2}} = \dfrac{31.5}{\frac{1}{2}} = 31.5 \cdot 2$

$\boxed{63 \text{ games}}$

8. If the sum of a geometric series is 101.01, and the first term is 100, and the last term is 0.01, find:
   i) the number of terms *ii) the common ratio*

$a = 100$ i) $S_n = \dfrac{a - rl}{1-r}$

$l = 0.01$

$S_n = 101.01$ $(1-r)\,101.01 = \dfrac{100 - 0.01r}{\cancel{1-r}}$ $\cancel{1-r}$

$101.01 - 101.01r = 100 - 0.01r$

$1.01 = 101.00r$

$\boxed{r = \dfrac{1}{100}} \longleftarrow r = \dfrac{1.01}{101} = 0.01$ or $\dfrac{1}{100}$

$t_n = ar^{n-1}$

$0.01 = 100\left(\dfrac{1}{100}\right)^{n-1}$

$0.0001 = \left(\dfrac{1}{100}\right)^{n-1}$

$\downarrow$

$\dfrac{1}{10\,000} = \dfrac{1}{100^{n-1}}$

$\dfrac{1}{100^2} = \dfrac{1}{100^{n-1}}$

$\boxed{n = 3}$

9. If you invest $1000 at the beginning of each year at 10% interest compounded annually, what is the value of the annuity at the end of 30 years? How much of the total is accumulated interest.

$a = 1000(1.1)$ ← 1000 times 110% to add interest to total

$n = 30$

$r = 1.1$

$S_{30} = 1000(1.1) + 1000(1.1)^2 + 1000(1.1)^3 + \ldots + 1000(1.1)^{30}$

$S_n = \dfrac{a(1-r^n)}{1-r} \Rightarrow S_{30} = \dfrac{1100(1-1.1^{30})}{1-1.1} \rightarrow \boxed{\$180\,943.43}$

10. Simplify $(1-r)(1+r+r^2+r^3+\cdots+r^{n-1})$

times all this minus r times all this

$$
\begin{array}{rl}
 & 1 + r + r^2 + r^3 + \ldots + r^{n-1} + 0 \\
- & \quad\;\; r + r^2 + r^3 + \ldots + r^{n-1} + r^n \\
\hline
 & 1 \quad \text{all cancels out except} \quad -r^n
\end{array}
$$

$\boxed{1 - r^n}$

11. You are offered two paychecks: \$40 000 with increases of 5% for 5 $years$, or \$43 000 with increases of 3% for 5 $years$. Which offer is better, and by how much, if:
    i. Your goal is to have the largest pay after 5 $years$ → after 5 yrs is $n=6$
    ii. Your goal is to have the largest total amount of money after 5 $years$.

$$S_n = \frac{a(1-r^n)}{1-r}$$

t_n Scenario

i) $t_n = ar^{n-1}$ this one

if $n=6$ $a=40000$ $r=1.05$

$$t_6 = 40000(1.05)^5 = \boxed{\$51\,051.26}$$

if $n=6$ $a=43000$ $r=1.03$

$$t_6 = 43000(1.03)^5 = \boxed{\$49848.79}$$

ii) if $n=5$, $a=40000$, $r=1.05$

$$S_5 = \frac{40000(1-1.05^5)}{1-1.05} = \boxed{\$221\,025.25}$$

$n=5$, $a=43000$, $r=1.03$

$$S_5 = \frac{43000(1-1.03^5)}{1-1.03} = \boxed{\$228\,292.84}$$

this one

12. If a person received a 10% salary increase each year and earned a total of \$155 680.05 by the end of the 5th year, determine the starting salary.

$$S_5 = \frac{a(1-r^n)}{1-r}$$

$S_5 = 155\,680.05$ $n=5$

$r = 110\% = 1.1$

$$155\,680.05 = \frac{a(1-(1.1)^5)}{1-1.1}$$

$$155\,680.05 = \frac{a(1-(1.1)^5)}{-0.1}$$

$$-15\,568.005 = a(1-(1.1)^5)$$

$$a = \frac{-15\,568.005}{1-(1.1)^5}$$

$$\boxed{a = \$25\,500}$$

13. An equilateral triangle has sides of length 10. If the midpoints of each side are joined to form another triangle, and this process is continued, what is:
    i) The perimeter of the $5th$ triangle?
    ii) The total perimeter of the first 5 triangles?

i) $n=5$

$a=30$

$r=\frac{1}{2}$

Length decreases by half every time

$P=30 \rightarrow 15 \rightarrow \dots$

$t_n = ar^{n-1}$

$t_5 = 30\left(\frac{1}{2}\right)^4$

$= 30\left(\frac{1}{16}\right)$

$= \frac{30}{16} = \boxed{\frac{15}{8}}$

ii) $S_n = \frac{a(1-r^n)}{1-r}$

$S_5 = \frac{30(1-(\frac{1}{2})^5)}{1-\frac{1}{2}}$

$= \frac{30(\frac{31}{32})}{\frac{1}{2}} = \boxed{58\frac{1}{8}}$

14. Terry Fox decided to walk $120km$ by walking 40% of the distance remaining each day. How far does he have remaining to walk after six days of walking?

If total is 120km

first day he walks $120 \cdot 0.40$

$= 48km$

$a = 48km$

$r = 0.40$

$n = 6$

$S_6 = \frac{48(1-(0.40)^6)}{1-0.40} = 79.7km$

So total remaining:

$120 - 79.7 = \boxed{40.3km}$

15. What is bigger and by how much?

    A) $1000 + 999 + 998 + \cdots; n = 1000$
    B) $1 + 2 + 4 + \cdots; n = 19$

A is common difference $S_n = \frac{n}{2}(2a + (n-1)d) \rightarrow S_{1000} = \frac{1000}{2}(2(1000) + (999)(-1))$

$= 500(1001)$

$= \boxed{500\,500}$

B is ratio $S_n = \frac{a(1-r^n)}{1-r} \rightarrow S_{19} = \frac{1(1-2^{19})}{1-2}$

$= \boxed{524\,287}$

Larger one is B by 23 787

16. The sum of the first and second term of a Geometric progression is 4 and the sum of the third and fourth term is 36. What is the first term?

$t_1 = a \quad t_2 = ar \quad t_3 = ar^2 \quad t_4 = ar^3$

$a + ar = 4$

$a(1+r) = 4$

$1 + r = \frac{4}{a}$

$ar^2 + ar^3 = 36$

$ar^2(1+r) = 36$

$1 + r = \frac{36}{ar^2}$

both equal $(1+r)$

$\frac{4}{a} = \frac{36}{ar^2}$

$\frac{4ar^2}{4a} = \frac{36a}{4a} \rightarrow r^2 = 9$

$\boxed{r = \pm 3}$

if $r = 3$ $\quad a(1+3) = 4$

$\boxed{a = 1}$

if $r = -3$ $\quad a(1-3) = 4$

$a(-2) = 4$

$\boxed{a = -2}$

17. Determine the second term of a Geometric Sequence if $a_4 + a_5 = -3$ and $a_3 + a_4 = -6$

$a_5 = a_4 r$

$a_4 + a_4 r = -3$

$a_4(1+r) = -3$

but $a_4 = a_3 r$

$\downarrow$

$a_3 r(1+r) = -3$

$\frac{-3}{a_3 r} = 1 + r$

$a_4 = a_3 r$

$a_3 + a_3 r = -6$

$a_3(1+r) = -6$

$1 + r = \frac{-6}{a_3}$

$\downarrow$

so $\frac{-3}{a_3 r} = \frac{-6}{a_3}$

$\frac{-3a_3}{-6} = \frac{-6a_3 r}{-6}$

$\frac{1}{2}a_3 = a_3 r$ so $\boxed{r = \frac{1}{2}}$

$a_3(1 + \frac{1}{2}) = -6$

$a_3(\frac{3}{2}) = -6 \rightarrow a_3 = -4$

so, $a_2 r = a_3$

$a_2(\frac{1}{2}) = -4$

$\boxed{a_2 = -8}$

18. If the sum of a Geometric Series is $S_n = 2(3^n - 1)$ then what is the fifth term of the series.

$S_n = 2(3^n - 1)$

$S_1 = 2(3^1 - 1)$

$S_1 = 4$

so

$a = 4$

$S_2 = 2(3^2 - 1)$

$S_2 = 16$

so

$a_1 + a_2 = 16$

$4 + a_2 = 16$

$a_2 = 12$

$r = \frac{12}{4} = 3$

$a = 4$

$r = 3$

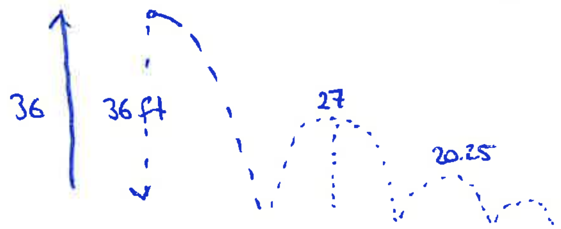
$n = 5$

$t_5 = ar^{n-1}$

$t_5 = 4(3)^4$

$= \boxed{324}$

19. A ball bounces up three-quarters of the distance form which it falls. How far has the ball travelled in total after hitting the floor the fourth time, if it is dropped from $36ft$.



Have to start on the floor so add 36 in distance to give up/down. But then subtract the added 36 at the end.

$a = 72$

$r = 0.75$

$n = 4$

$$S_n = \frac{a(1-r^n)}{1-r} - 36$$

$$S_4 = \frac{72(1-0.75^4)}{1-0.75} - 36$$

$$= 196.875 - 36$$

$$= \boxed{160.88\,ft}$$

20. Can you solve this riddle:

As I was going to St. Ives 1
I met a man with seven wives $1+7$
Every wife had seven sacks 7^2
Every sack had seven cats 7^3
Every cat had seven kits 7^4
Kits, cats, sacks, and wives,
How many were going to St Ives?

If we assume they were all going to St. Ives

$1+1+7+7^2+7^3+7^4$

$\boxed{2802}$

But there is no direct mention of whether everyone was going, only "I was"

$\boxed{1}$

See Website for Detailed Answer Key