Section 1.3 – Geometric Series

- Just like how the summation of an **Arithmetic Sequences** is an **Arithmetic Series**
- The summation of a **Geometric Sequence** is a **Geometric Series**
- If the pattern is **finite (ends)** then we have **a Finite Geometric Series**
- If the pattern is **infinite,** then guess what, we have an **Infinite Geometric Series (Section 1.4)**

Geometric Series

The sum of the first n terms of a geometric series with first term a and last term l is given by:

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a-ar^n}{1-r}; \quad or \quad \frac{a-rl}{1-r}, for\ r \neq 1$$

Example 1: Find the sum of the Geometric Series $2 + 6 + 18 + 54 + \cdots + 1458$

Finite we have a last term l

Solution 1:

Since we do not know n, we have to use the equation with l

$$S_n = \frac{a-rl}{1-r}; \quad a = 2, r = 3, l = 1458$$

$$S_n = \frac{2-(3)(1458)}{1-3} \rightarrow \frac{-4372}{-2} = 2186$$

Example 2: Find the sum of the first seven terms of the Geometric Series $3 + 6 + 12 + \cdots$

Solution 2: Since we do not know l, we use the equation without l

$$S_n = \frac{a(1-r^n)}{1-r}; \quad a = 3, r = 2, n = 7$$

$$S_n = \frac{3(1-2^7)}{1-2} \rightarrow \frac{-381}{-1} = 381$$

Example 3: Find the sum of the Geometric Series

$$\sum_{k=1}^{10} 3(-2)^{k-1}$$

Solution 3:

We can use either equation here, but we need to think about which number means what.

Number of terms → $\sum_{k=1}^{10} 3(-2)^{k-1}$ ← Common Ratio

We get a when we set $k = 1$ $\quad a = 3(-2)^{1-1} \rightarrow 3(-2)^0 \rightarrow a = 3$

We get l when we set $k = 10$ $\quad l = 3(-2)^{10-1} \rightarrow 3(-2)^9 \rightarrow l = -1536$

Now we can use either equation.

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{3(1 - (-2)^{10})}{1 - (-2)} \rightarrow \frac{-3069}{3} = -1023$$

$$S_n = \frac{a - rl}{1 - r} = \frac{3 - (-2)(-1536)}{1 - (-2)} \rightarrow \frac{-3069}{3} = -1023$$

Example 4: Write the Geometric Series, $6 + 18 + 54 + 162 + 486$, using sigma notation with index $k = 1$

Solution 4: $n = 5, r = 3$

In a Geometric Series the ***r value*** is written to a **variable exponent,** $\boldsymbol{r = 3} \rightarrow \sum_{k=1}^{5} 3^k$

To get the initial ***a value*** the ***r value*** was **multiplied by 2**.

So… $\sum_{k=1}^{5} 2 \cdot 3^k$

Note: In Summation Notation

An **Arithmetic Series** has the **variable in the base**. $\sum_{k=1}^{5} 2k - 3$

A **Geometric Series** has its **variable in the exponent**. $\sum_{k=1}^{5} 2 \cdot 3^k$

Recall:

Sigma Notation has

$$n - k + 1 \; Terms$$

Section 1.3 – Practice Problems

1. Determine the desired information.

a) $S_{10,}\ if\ a = 8, r = \frac{1}{2}$

b) $S_{9,}\ if\ a = -6, r = 2$

c) $a, if\ S_8 = 765, r = 2$

d) $S_{6,}\ if\ a = -8,\ t_4 = 27$

e) $S_{5,}\ if\ t_3 = 3, r = \frac{1}{2}$

f) $S_{8,}\ if\ a = 12, t_5 = 192$

g) $t_{3,}\ if\ S_5 = 93, r = 2$

h) $r, if\ a = 3, S_3 = 39$

i) $S_{100},\ if\ \ t_1 = -1, t_2 = 1, t_3 = -1, t_4 = 1$

j) $S_{101},\ if\ \ t_1 = -1, t_2 = 1, t_3 = -1, t_4 = 1$

2. Find the indicated value using the information given.

a) $\sum_{k=8}^{35}(2^k - 5)$

b) $\sum_{i=8}^{b}(2^i - 5)$

c) $\sum_{k=a}^{9}(2^k - 5)$

d) $\sum_{i=a}^{b}(2^i - 5)$

3. Find the sum of each Geometric Series

. $\sum_{k=1}^{8} 3 \cdot (2)^{k-1}$

. $\sum_{k=1}^{12} 2 \cdot (-3)^{k-1}$

. $\sum_{x=0}^{10} 5 \cdot \left(\frac{1}{2}\right)^{x}$

. $\sum_{k=0}^{9} \frac{3}{5^{k+1}}$

. $\sum_{b=2}^{9} 18 \cdot (0.1)^{b}$

. $\sum_{i=3}^{11} \frac{2^{i}}{3^{i-1}}$

. $\sum_{k=1}^{n} \left(\frac{2}{3}\right)^{k}$

. $\sum_{k=1}^{n} 4 \cdot 3^{k-1}$

4. Write the Geometric Series using Sigma Notation with $k = 1$ as the index of the first term.

a) $3 + 6 + 12 + 24 + 48$

b) $2 - 6 + 18 - 54 + \cdots + 13\,122$

c) $\frac{3}{16} - \frac{3}{8} + \frac{3}{4} - \cdots - 384$

d) $8 + 4 + 2 + 1 + \cdots + \frac{1}{1024}$

5. Solve for a: $\sum_{i=0}^{2} a^i = 31$

6. Which is larger, and by what amount? $2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$, or 2^n

7. If 64 students enter a singles tennis tournament, where the winner of each match advances to the next round, how many matches must be played before a winner is determined?

8. If the sum of a geometric series is 101.01, and the first term is 100, and the last term is 0.01, find:
   $i)\ the\ number\ of\ terms \qquad ii)\ the\ common\ ratio$

9. If you invest $\$1000$ at the beginning of each year at 10% interest compounded annually, what is the value of the annuity at the end of 30 years? How much of the total is accumulated interest.

10. Simplify $(1-r)(1+r+r^2+r^3+\cdots+r^{n-1})$

11. You are offered two paychecks: $\$40\ 000$ with increases of $5\%\ for\ 5\ years$, or $\$43\ 000$ with increases of $3\%\ for\ 5\ years$. Which offer is better, and by how much, if:
    i. Your goal is to have the largest pay after $5\ years$
    ii. Your goal is to have the largest total amount of money after $5\ years$.

12. If a person received a 10% salary increase each year and earned a total of $\$155\ 680.05$ by the end of the $5th$ year, determine the starting salary.

13. An equilateral triangle has sides of length 10. If the midpoints of each side are joined to form another triangle, and this process is continued, what is:
    i) The perimeter of the $5th$ triangle?
    ii) The total perimeter of the first 5 triangles?

14. Terry Fox decided to walk $120km$ by walking 40% of the distance remaining each day. How far does he have remaining to walk after six days of walking?

15. What is bigger and by how much?

    A) $1000 + 999 + 998 + \cdots ; n = 1000$
    B) $1 + 2 + 4 + \cdots ; n = 19$

16. The sum of the first and second term of a Geometric progression is 4 and the sum of the third and fourth term is 36. What is the first term?

17. Determine the second term of a Geometric Sequence if $a_4 + a_5 = -3$ and $a_3 + a_4 = -6$

18. If the sum of a Geometric Series is $S_n = 2(3^n - 1)$ then what is the fifth term of the series.

19. A ball bounces up three-quarters of the distance form which it falls. How far has the ball travelled in total after hitting the floor the fourth time, if it is dropped from $36ft$.

20. Can you solve this riddle:

$As\ I\ was\ going\ to\ St.\ Ives$
$I\ met\ a\ man\ with\ seven\ wives$
$Every\ wife\ had\ seven\ sacks$
$Every\ sack\ had\ seven\ cats$
$Every\ cat\ had\ seven\ kits$
$Kits, cats, sacks, and\ wives,$
$How\ many\ were\ going\ to\ St\ Ives?$

See Website for Detailed Answer Key

Extra Work Space