Section 1.2 – Geometric Sequence

- Whereas **Arithmetic Sequences** involve a **Common Difference** (Addition/Subtraction)
- A **Geometric Sequence** has a **Common Ratio** (Multiplication/Division)

Consider this sequence…

$$\mathbf{3, 6, 12, 24, \ldots,}$$

- $6 \div 3 = 2$
- $12 \div 6 = 2$
- $24 \div 12 = 2$

$\boxed{The\ \boldsymbol{Common\ Ratio}\ is\ 2}$

Geometric Sequence

A sequence is Geometric when the $\boldsymbol{common\ ratio}$ $(\boldsymbol{r})$ is constant.

The $r, where\ \ r \neq 0\ is$: $\dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \dfrac{a_n}{a_{n-1}} = r$

Example 1: Find the Common Ratio of the following sequences

a) $2, 6, 18, 54, \ldots$
b) $3, -6, 12, -24, \ldots$
c) $-8, -4, -2, -1, \ldots$

Solution 1:

a) $\dfrac{6}{2} = \dfrac{18}{6} = \dfrac{54}{18} = 3 \quad \rightarrow \quad r = 3$

b) $\dfrac{-6}{3} = \dfrac{12}{-6} = \dfrac{-24}{12} = r \quad \rightarrow \quad r = -2$

c) $\dfrac{-4}{-8} = \dfrac{-2}{-4} = \dfrac{-1}{-2} = r \quad \rightarrow \quad r = \dfrac{1}{2}$

Much like the formula for an Arithmetic Sequence, we can derive the Formula for a Geometric Formula using **the Common Ratio Logic**.

Let $a = the\ first\ term\ and\ r = the\ common\ ratio$

$$a_1$$

$$a_2 = a_1 \cdot r$$

$$a_3 = a_2 \cdot r = (a_1 \cdot r) \cdot r = a_1 \cdot r^2$$

$$a_4 = a_3 \cdot r = (a_1 \cdot r^2) \cdot r = a_1 \cdot r^3$$

$$\vdots$$

$$\boldsymbol{a_n = a_1 \cdot r^{n-1}}$$

The n^{th} term of a Geometric Sequence

$\boldsymbol{t_n = ar^{n-1}}$, for any integer $n \geq 1$

Example 2: Find the 10th term of the following Geometric Sequence $3, 12, 48, 192, \ldots$

Solution 2: The Common Ratio is: $12 \div 3 = \mathbf{4}$**, with** $\boldsymbol{a = 3}$

$$t_n = ar^{n-1} \quad \rightarrow \quad t_{10} = (3)(4)^{10-1}$$

$$t_{10} = (3)(4)^9 \quad \rightarrow \quad t_{10} = (3)(262\,144)$$

$$\boldsymbol{t_{10} = 786\,432}$$

Example 3: The 4th term of a Geometric Sequence is 125, and the 9th term is $\frac{125}{32}$. Find the 13th term.

Solution 3: The 4th term $\boldsymbol{t_4 = ar^3 = 125}$. The 9th term $\boldsymbol{t_9 = ar^8 = \frac{125}{32}}$

We can write $\boldsymbol{ar^8}$ as $(\boldsymbol{ar^3}) \cdot \boldsymbol{r^5}$ **(remember your exponent laws)**

So… $ar^8 = (ar^3) \cdot r^5 = \frac{125}{32} \quad \rightarrow \quad 125 \cdot r^5 = \frac{125}{32} \quad \rightarrow \quad r^5 = \frac{1}{32}$

Sub in

Divide both sides by 125

Fifth-root both sides

$$r = \sqrt[5]{\frac{1}{32}} = \frac{\sqrt[5]{1}}{\sqrt[5]{32}} = \frac{1}{2}$$

Now that we have r, we can solve for a

$$ar^3 = 125 \quad \rightarrow \quad a\left(\frac{1}{2}\right)^3 = 125 \quad \rightarrow \quad a\left(\frac{1}{8}\right) = 125 \quad \rightarrow \quad a = 1000$$

And now we can solve for any term we want.

$$t_{13} = ar^{13-1} \quad \rightarrow \quad t_{13} = 1000\left(\frac{1}{2}\right)^{12} \quad \rightarrow \quad \boldsymbol{t_{13} = \frac{125}{512}}$$

Example 4: What is the value of x in the sequence: $x, 2x + 2, 3x + 3$

Solution 4: We need to write the statements as ratios

$$\frac{a_2}{a_1} = r \qquad and \qquad \frac{a_3}{a_2} = r$$

$$\frac{2x + 2}{x} = r \qquad and \qquad \frac{3x + 3}{2x + 2} = r$$

Since **both sides are equal to $\boldsymbol{r}$** we can set them equal to each other.

$$\frac{2x + 2}{x} = \frac{3x + 3}{2x + 2}$$

Multiply both sides of the equation by the LCM.

$x(2x + 2)$

Cross Multiply

FOIL and Waterbomb

$$(2x + 2)(2x + 2) = (3x + 3)(x)$$

$$4x^2 + 8x + 4 = 3x^2 + 3x$$

$$x^2 + 5x + 4 = 0$$

Isolate the Quadratic and Solve

$$(x + 4)(x + 1) = 0$$

$$x = -1 \; and -4$$

Section 1.2 – Practice Problems

1. Determine if the sequence is geometric. If so, find r.

a) $4, 12, 36, 72, \ldots$

b) $3, 12, 48, 142, \ldots$

c) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

d) $1, -1, 1, -1, \ldots$

e) $3, -6, -12, 24, \ldots$

f) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

g) $\frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \frac{2}{27}, \ldots$

h) $\frac{2}{5}, -\frac{2}{3}, \frac{10}{9}, -\frac{50}{27}, \ldots$

i) $3x^2, 12x^4y^3, 48x^6y^6, \ldots$

j) $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, 3\sqrt{6}, \ldots$

2. Write the first five terms of the Geometric Sequence

a) $1, 4,$ ___, ___, ___

b) $1,$ ___, $4,$ ___, ___

c) $\frac{1}{2},$ ___, ___, $\frac{1}{16},$ ___

d) $4,$ ___, ___, $-13.5,$ ___

e) ___, $54, 18,$ ___, ___

f) $1,$ ___, $3,$ ___, ___

g) $3,$ ___, $3^{2x+1},$ ___, ___

h) $1,$ ___, $x^4,$ ___, ___

i) $5, 5^{2x-1},$ ___, ___, ___

j) $1, -\frac{x}{3},$ ___, ___, ___

3. Find all possible values of r for a Geometric Sequence with the two given terms.

a) $a_5 = 5, a_7 = 25$

b) $a_2 = 4, a_6 = \frac{1}{4}$

c) $a_4 = 2\sqrt{2}, a_7 = 8$

d) $a_3 = 1, a_6 = \sqrt{2}$

4. Find the desired information.

a) $a_{11},\ if\ \ a_1 = \frac{1}{128}, r = 2$

b) $a_9,\ if\ a_1 = 3, a_2 = \sqrt{3}$

c) $a_{42},\ if\ a_{40} = 9,\ a_{41} = 36$

d) $a_9,\ if\ a_4 = 5, a_6 = 20$

e) $n,\ if\ a_1 = 729, a_2 = 243, l = \frac{1}{9}$

f) $n,\ if\ a_1 = 2048, a_2 = 1024, l = 1$

g) $a_1,\ if\ \ a_5 = 27, r = 3$

h) $a_1,\ if\ \ a_7 = 128, r = 4$

i) $r,\ if\ \ a_{10} = 25,\ a_{12} = 225$

j) $r,\ if\ \ a_{25} = 12,\ a_{31} = 96$

k) $a_8,\ if\ \ a_n = 3a_{n-1}, a_1 = \frac{1}{27}$

l) $a_6,\ if\ \ a_n = 0.1a_{n-1}, a_1 = 1000$

5. Insert two Geometric Means between $a \; and \; b$.

6. Given the geometric sequence $a, \frac{a}{b}, \frac{a}{b_2}, \ldots$ determine an expression for $t_n - t_{n-1}, n > 2$

7. Find x so that $x - 1,\; x, and\; x + 2$ are consecutive (one after the other) terms of a Geometric Sequence.

8. Find the common ratio r for the geometric sequence: $x-2, 5-x, 5x-7, \ldots$

9. What number must be added to $-2, 4, 19$ so that the resulting numbers are three terms of a Geometric Sequence.

10. If the first two terms of a geometric sequence are $\sqrt{2},\ and\ \sqrt[3]{2},$ what is the fourth term?

11. If the product of the first three terms of a Geometric Sequence is -8, and the sum is $\frac{14}{3}$**,** what is the common ratio of the sequence?

12. In the sequence $3, x, y, 25$, the first three terms form an arithmetic sequence, and the last three terms form a geometric sequence. Find $x\ and\ y$.

13. The enrolment at Vic High in Victoria was $400\ in\ 1973$. If the school's population has increased by 5% a year, how many students will be going to the school in 2010.

14. If a starting salary is $\$28\ 000$ and you get an annual increase of 6%, what is your salary at the beginning of the eighth year of work?

15. With each cycle, a vacuum pump removes 25% of the air in a glass container. What percent of the air has been removed after $10\ cycles$.

16. A car costs a company $\$40\ 000$. Each year, the car depreciates 16% of its value. What is the value of the care after five years.

17. Initially a pendulum swings through an arc of $45cm$. On each successive swing, the length of the arc is decreased by 2% of the previous length. What is the length of the arc after $12\ swings$.

18. A ball is dropped from a height of $10\ meters$. Each time it strikes the ground it bounces up 75% of its previous height. How many bounces does the ball need before the bounce is less than 20 cm high (Watch your units)?

19. From farmer to consumer a shipment goes through a number of handlers: $Farmer \quad \rightarrow \quad Trucker \quad \rightarrow \quad Regional\ Market \quad \rightarrow \quad Trucker \quad \rightarrow \quad Wholesaler \quad \rightarrow \quad Trucker \rightarrow Retailer \quad \rightarrow \quad Consumer$. If the farmer get 75 $cents\ per\ kilogram$, and if each person in the chain makes a $20\%\ profit$, how much does the consumer pay?

20. A truck radiator contains fifty litres of water. Five litres of water is removed and replaced with pure antifreeze, then five litres of mixture is removed and replaced. How much antifreeze is in the radiator after the process is repeated five times.

See Website for Detailed Answer Key

Extra Work Space