Section 1.1 – Arithmetic Sequence and Series

- Depending on your experience, you may have seen this is Grade 10, but here is a quick review

A Sequence

- A sequence is a list of numbers
- A **finite sequence** stops $1, 3, 5, 7, 9$
- An **infinite sequence** does not $1, 3, 5, 7, 9, \ldots$

Each sequence is made of **terms,** and they have a **Domain of positive integers.**

The generic form of a sequence is in numbered terms…

$$\boldsymbol{a_1, a_2, a_3, \ldots, a_n}$$

a_n usually denotes an entire sequence

Example 1: Write the first four terms of the sequence

a) $a_n = \frac{n+1}{n}$ b) $b_n = 2n - 3$

Solution 1:

a) $a_1 = \frac{1+1}{1} = 2,\quad a_2 = \frac{2+1}{2} = \frac{3}{2},\quad a_3 = \frac{3+1}{3} = \frac{4}{3},\quad a_4 = \frac{4+1}{4} = \frac{5}{4}$

b) $b_1 = 2(1) - 3 = -1,\quad b_2 = 2(2) - 3 = 1,\quad b_3 = 2(3) - 3 = 3,\quad b_4 = 2(4) - 3 = 5$

- We can also define a sequence by identifying the first term, or first few terms, and then denoting the n^{th} term in terms of the preceding term, using the notation: a_{n-1}

This is called a $\boldsymbol{RECURSIVE\ SEQUENCE}$

Example 2: Write the first four terms of the recursive formula: $a_1 = 3,\quad a_n = \frac{a_{n-1}}{n}$

Solution 2:

$$a_1 = 3,\quad a_2 = \frac{a_{2-1}}{2} = \frac{a_1}{2} = \frac{3}{2},\quad a_3 = \frac{a_{3-1}}{2} = \frac{a_2}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2},\quad a_4 = \frac{a_{4-1}}{4} = \frac{a_3}{4} = \frac{\frac{1}{2}}{4} = \frac{1}{8}$$

Sigma Notation

There is a lovely, but strange looking notation that can **collapse the written form of a summed sequence**. It is denoted by the Greek Letter Sigma (Σ).

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Sigma Notation has

$$n - k + 1 \, Terms$$

- $\boldsymbol{k}$ is the **index of the sum**, and shows where the sum **starts**
- $\boldsymbol{n}$ shows where the **summation ends**

Example 3: Find the sum of the following sequences

$$\sum_{k=1}^{4}(2k+1) \qquad and \qquad \sum_{k=1}^{5}(k^2+1)$$

Solution 3:

For the first one, we **start at** $\boldsymbol{k = 1}$ and $\boldsymbol{finish\ at\ k = 4}$

$$2(1)+1=3, \quad 2(2)+1=5, \quad 2(3)+1=7, \quad 2(4)+1=9$$

$$3+5+7+9=24$$

For the second one, we **start at** $\boldsymbol{k = 1}$ and $\boldsymbol{finish\ at\ k = 5}$

$$1^2+1=2, \quad 2^2+1=5, \quad 3^2+1=10, \quad 4^2+1=17, \quad 5^2+1=26$$

$$2+5+10+17+26=60$$

Arithmetic Sequence

- An **arithmetic sequence** is a **sequence in which the successive terms** have a **common difference**
- For example the sequence, $3, 7, 11, 15, \ldots$ has a **common difference** of 4
- The common difference, $\boldsymbol{d}$, of this sequence is 4.

If we look at the pattern we may see something helpful…

We are able to express every term with respect to the first term and the common difference!

1st term: $a_1 = a_1$

2nd term: $\boldsymbol{a_2 = a_1 + d}$

3rd term: $\boldsymbol{a_3 = a_2} + d = (\boldsymbol{a_1 + d}) + d = \boldsymbol{a_1 + 2d}$

4th term: $\boldsymbol{a_4 = a_3} + d = (\boldsymbol{a_1 + 2d}) + d = \boldsymbol{a_1 + 3d}$

- From this pattern we are able to generate the general equation of an Arithmetic Sequence

The $\boldsymbol{n^{th}}$ term of an Arithmetic Sequence

- For an arithmetic sequence $\{\boldsymbol{t_n}\}$ whose first term is $\boldsymbol{a}$, with a common difference $\boldsymbol{d}$:

$$\boldsymbol{t_n = a + (n-1)d} \qquad \text{for any integer } n \geq 1$$

Example 4: For each arithmetic sequence, identity the common difference.

a) $3, 5, 7, 9, \ldots$
b) $11, 8, 5, 2, \ldots$

Solution 4:

a) $5 - 3 = 2$, $7 - 5 = 2$, $9 - 7 = 2$, Therefore $d = 2$
b) $8 - 11 = -3$, $5 - 8 = -3$, $2 - 5 = -3$, Therefore $d = -3$

Example 5: Determine if the sequence $\{t_n\} = \{3 - 2n\}$ is arithmetic

Solution 5:

$t_1 = 3 - 2(1) = 1$

$t_2 = 3 - 2(2) = -1$

$t_3 = 3 - 2(3) = -3$

$1, -1, -3, \ldots$ has a common difference of -2

So the sequence is arithmetic!

Example 6: Find the 12th term of the arithmetic sequence $2, 5, 8, \ldots$

Solution 6:

$a = 2 \qquad d = 3$

$t_n = a + (n-1)d$

$t_{12} = 2 + (12 - 1)3 \qquad \rightarrow \qquad 35$

Example 7: Which term in the arithmetic sequence $4, 7, 10, \ldots$ has a value of 439?

Solution 7:

$$d = 7 - 4 = 3$$

$$t_n = a + (n-1)d$$

$$439 = 4 + (n-1)3$$

$$435 = (n-1)3 \quad \rightarrow \quad 145 = n - 1$$

$$n = 146$$

The 146th term is 439.

Example 8: The 7th term of an arithmetic sequence is 78, and the 18th term is 45. Find the 1st term.

Solution 8: There are $18 - 7 = \mathbf{11}$ **terms** between 45 and 78. And the difference between them is $\mathbf{45 - 78 = -33}$

So, $11d = -33 \quad \rightarrow \quad d = -3$

This gets us the common difference by using the gap in all the terms we missed.

So,

$$t_n = a + (n-1)d$$

$$t_7 = a + (7-1)(-3)$$

$$78 = a + (-18)$$

$$78 + 18 = a = 96$$

Can use the 7th term or 18th

Example 9: Find x so that $3x + 2, 2x - 3, and\ 2 - 4x$ are consecutive terms of an arithmetic sequence

Solution 9: Since they are consecutive,

$$(2x - 3) - (3x + 2) = d \qquad \text{and} \qquad (2 - 4x) - (2x - 3) = d$$

So since they both equal d, we can set them equal to each other.

$$(2x - 3) - (3x + 2) = (2 - 4x) - (2x - 3)$$

$$2x - 3 - 3x - 2 = 2 - 4x - 2x + 3$$

$$-x - 5 = -6x + 5 \quad \rightarrow \quad 5x = 10 \quad \rightarrow \quad \boldsymbol{x = 2}$$

Arithmetic Series

- An **arithmetic series** is when we take our given **sequence** and we **add it all together** (sum)
- We have **finite and infinite sums** just like we have for sequences, but we're only going to look at **finite series**
- Here's the formula:

Sum of an Arithmetic Series

- The sum of the first n terms of an arithmetic series is given by:

$$S_n = \frac{n}{2}(a + l) \qquad or \qquad S_n = \frac{n}{2}(2a + (n-1)d)$$

Where $\boldsymbol{a = the\ first\ term,}$ $\boldsymbol{l = the\ last\ term,}$ **and** $\boldsymbol{d = the\ common\ difference}$

- We can interchange the two equations, depending on what information is given to us
- Then it really just becomes plug by numbers

Example 10: Find the sum of the positive integers from $1\ to\ 50$ inclusive.

Solution 10:

$$a = 1, \quad l = 50, \quad d = 1$$

$$S_n = \frac{n}{2}(a + l)$$

Since we have $\boldsymbol{a\ and\ l}$ we know we can use this one.

$$S_{50} = \frac{50}{2}(1 + 50) \quad \rightarrow \quad 25(51) \quad \rightarrow \quad 1275$$

Example 11: Find the sum of the first 25 terms of the series $11 + 15 + 19 + \cdots$

Solution 11: The series is arithmetic (has a common difference) with $a = 11, d = 4, and\ n = 25$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Since we **don't know** $\boldsymbol{l}$ we know we have to use this one.

$$S_{25} = \frac{25}{2}(2(11) + (25-1)4) \quad \rightarrow \quad 12.5(22 + 96) \quad \rightarrow \quad 1475$$

Example 12: Find the sum of the series $7 + 10 + 13 + \cdots + 100$.

Solution 12: $a = 7, l = 100, d = 3$ but We **don't know n**, so we solve for that first

To **find n** we use the **formula from Section 2.2**

$$t_n = a + (n-1)d$$

Since we want to know **how many terms** there are and **100 is the last term**, if we solve for that we'll get n.

$$100 = 7 + (n-1)(3)$$
$$\to 100 = 7 + 3n - 3$$
$$\to 100 = 4 + 3n$$
$$\to 96 = 3n$$
$$n = 32$$

Now we can solve for the sum since we know n

$$S_n = \frac{n}{2}(a + l)$$

$$S_{32} = \frac{32}{2}(7 + 100)$$

$$S_{32} = 16(107)$$

$$S_{32} = 1712$$

Example 13: Find the sum of the $5 + 9 + 13 + \cdots + 137$

Solution 13: $a = 5, l = 137, d = 4$ but We don't know n, so we solve for that first

To **find n** we use the **formula from Section 2.2**

$$t_n = a + (n-1)d$$

Since we want to **know how many terms** there are and **137 is the last term**, if we solve for that we'll get n.

$$137 = 5 + (n-1)(4)$$
$$\to 137 = 5 + 4n - 4$$
$$\to 137 = 1 + 4n$$
$$\to 136 = 4n$$
$$n = 34$$

Now we can solve for the sum since we know n

$$S_n = \frac{n}{2}(a + l)$$

$$S_{34} = \frac{34}{2}(5 + 137)$$

$$S_{34} = 17(142)$$

$$S_{34} = 2414$$

Section 1.1 Part 1 – Practice Problems

a) Respond to the following

a) The Domain of a sequence is the set of what find of consecutive numbers?

b) A sequence with a last term is what kind of sequence?

c) A sequence with no last term is what kind of sequence

d) What kind of sequence is this: $a_1 = 2, \quad a_n = 2a_{n-1}$

e) Write the formula for the n^{th} term of an arithmetic sequence t_n

2. Write the first four terms of each of the following sequences

a) $\{n^2 - 2\}$

b) $\left\{\frac{n+2}{n+1}\right\}$

c) $\{(-1)^{n+1}n^2\}$

d) $\left\{\frac{3^n}{2^n+1}\right\}$

e) $\left\{\frac{2^n}{n^2}\right\}$

f) $\left\{\left(\frac{2}{3}\right)^n\right\}$

3. Write the n^{th} term of the suggested pattern.

a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

b) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

c) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

d) $2, -4, 6, -8$

4. Write the first four terms of the recursive sequence.

a) $a = 4, t_n = 2 + t_{n-1}$

b) $a = 3, t_n = n - t_{n-1}$

c) $a = 2, a_2 = 3, a_n = a_{n-1} + a_{n-2}$

d) $a = -1, a_2 = 1, a_n = na_{n-1} + a_{n-2}$

5. Find the sum of each sequence.

a) $$\sum_{k=1}^{5} 4$$

b) $$\sum_{k=1}^{4} (k^2 - 2)$$

c) $$\sum_{k=2}^{5} (k^2 - 1)$$

d) $$\sum_{k=0}^{3} (k^3 - 1)$$

e) $$\sum_{k=1}^{4} \frac{k^2}{2}$$

f) $$\sum_{k=6}^{8} (k + 1)^2$$

6. Express each sum using summation notation with index $k = 1$

a) $1 + 3 + 5 + 7$

b) $1^2 + 2^2 + 3^2 + 4^2 + 5^2$

c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1}$

d) $5 + \frac{5^2}{2} + \frac{5^3}{3} + \cdots + \frac{5^n}{n}$

7. Write the first five terms of the arithmetic sequence.

a) $7, 11, 15,$ ____, ____

b) $15, 12, 9,$ ____, ____

c) $a = 4, d = 2$

d) $a = -1, d = -3$

e) $a = -5, d = -\frac{3}{4}$

f) $a = -\frac{2}{3}, d = \frac{1}{5}$

8. Find the indicated arithmetic term.

a) $a = 5, d = 3; find\ t_{12}$

b) $a = \frac{2}{3}, d = -\frac{1}{4}; find\ t_9$

c) $a = -\frac{3}{4}, d = \frac{1}{2}; find\ t_{10}$

d) $a = 2.5, d = -1.253; find\ t_{20}$

e) $a = -0.75, d = 0.05; find\ t_{40}$

f) $a = -\frac{7}{4}, d = -\frac{2}{3}; find\ t_{37}$

9. Find the number of terms in each arithmetic sequence

a) $a = 6, d = -3,\ t_n = -30$

b) $a = -3, d = 5,\ t_n = 82$

c) $a = 0.6, d = 0.2, t_n = 9.2$

d) $a = -0.3, d = -2.3,\ t_n = -39.4$

e) $-1, 4, 9, \ldots, 159$

f) $23, 20, 17, \ldots, -100$

10. Find the first term in the arithmetic sequence

a) $6th\ term\ is\ 10; 18th\ term\ is\ 46$

b) $4th\ term\ is\ 2; 18th\ term\ is\ 30$

c) $9th\ term\ is\ 23; 17th\ term\ is - 1$

d) $5th\ term\ is\ 3; 25th\ term\ is - 57$

e) $13th\ term\ is-3; 20th\ term\ is-17$

f) $11th\ term\ is\ 37; 26th\ term\ is 32$

11. Find x so that the values given are consecutive terms of an arithmetic sequence

a) $x+3, 2x+1, and\ 5x+2$

b) $2x, 3x+2, and\ 5x+3$

c) $x-1, \frac{1}{2}x+4, and\ 1-2x$

d) $2x-1, x+1, and\ 3x+9$

e) $x+4, x^2+5, and\ x+30$

f) $8x+7, 2x+5, and\ 2x^2+x$

12. If t_n is a term of an arithmetic sequence, what is $t_n - t_{n-1}$ equal to?

13. List the first seven numbers of the Fibonacci Sequence: $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}, n > 2$

14. The starting salary of an employee is $\$23\ 750$. If each year a $\$1250$ raise is given, in how many years will the employee's salary be $\$50\ 000$?

15. An auditorium has 8 seats in the first row. Each subsequent row has 4 more seats than the previous row. What row has 140 seats?

16. A well drilling company charges $\$8.00$ for the first meter, then $\$8.75$ for the second meter and so on in an arithmetic sequence. At this rate, what would be the cost to drill the meter of a well $120\ meters$ deep?

17. Assume that leading up to your death you needed $15\ minutes$ more sleep each night. When you hit an additional $24\ hours$ sleep you die. If you needed $8\ hours$ sleep on September 1st, what day do you die?

18. The first three terms of an arithmetic sequence are: $x - 3, \frac{x^2}{25} + 9, and\ 3x - 11$. Determine the fourth term.

19. The first, third, and fifth terms of an arithmetic sequence are: $2x - 1, x^2 - 3, and\ 11 - x^2$ respectively. Determine the second term.

Section 1.1 Part 2 – Practice Problems

1. Find the sum of the arithmetic series

a) $3 + 5 + 7 + \cdots + (2n + 1)$

b) $-1 + 2 + 5 + \cdots + (3n - 4)$

c) $2 + 5 + 8 + \cdots + 77$

d) $5 + 9 + 13 + \cdots + 97$

e) $(-41) + (-35) + (-29) + \cdots + 541$

f) $2\sqrt{5} + 6\sqrt{5} + 10\sqrt{5} + \cdots + 50\sqrt{5}$

g) $39 + 33 + 27 + \cdots + (-15)$

h) $23 + 19 + 15 + \cdots + (-305)$

i) $\frac{1}{2} + \frac{7}{8} + \frac{5}{4} + \cdots + \frac{55}{8}$

j) $\frac{16}{3} + \frac{13}{3} + \frac{10}{3} + \cdots + \left(-\frac{65}{3}\right)$

2. Find the indicated value using the information given

a) $S_{20}, if\ a_1 = 8, a_{20} = 65$

b) $S_{21}, if\ a_1 = 8, a_{20} = 65$

c) $S_{56}, if\ a_{56} = 13, d = -9$

d) $n\ if\ S_n = 180, a_1 = 4, t_n = 16$

e) $d, if\ S_{40} = 680, a_1 = 11$

f) $S_{62}, if\ a_1 = 10, d = 3$

g) $S_{19}, if\ d = 4, a_{19} = 17$

h) $S_{40}, if\ d = -3, a_{40} = 65$

i) $S_{40}, if\ a_5 = 42, a_{15} = -18$

j) $S_{20}, if\ a_8 = 17, a_{15} = 38$

3. Find the indicated sum.

a) $\sum_{n=1}^{100} n$

b) $\sum_{k=100}^{200} k$

c) $\sum_{j=0}^{72} (3j - 4)$

d) $\sum_{x=7}^{24} (2x + 5)$

e) $\sum_{y=11}^{48}\left(\frac{y+4}{2}\right)$

f) $\sum_{z=51}^{100}(200-z)-\sum_{z=1}^{50}(200-z)$

4. Insert k arithmetic means between the given pair of numbers.

a) $5, 10, k = 2$

b) $3, 6, k = 3$

c) $a, b, k = 2$

d) $a, b, k = 3$

5. Solve for b: $\sum_{x=2}^{b}(23 - 2x) = 91$

6. Find the sum: $\sum_{x=a}^{b} 5$

7. What is the last element in the 20^{th} row?

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 7 & & 8 & & 9 & & 10 \end{array}$$

8. How many terms of the arithmetic series $1491 + 1484 + 1477 + \cdots$ are needed to give the sum of zero?

9. An auditorium has eight seats in the first row. Each subsequent row has four more seats than the previous row. How many seats are there in the 50^{th} row of the auditorium?

10. If $\$1000$ is deposited into the bank the day a child is born, and $\$100$ more than the previous deposit is made each year until the child's $18th$ birthday, how much will be in the account, excluding interest?

11. Find the sum of all multiples of 6 between $50 \ and \ 500$.

12. The sum of three consecutive terms of an arithmetic sequence is 3. The sum of their squares is 75. Find the three numbers.

13. If 20 people in a class shake hands with each other exactly once, how many handshakes will take place?

14. If the sum of the terms of an arithmetic series is 234, and the middle term is 26, find the number of terms in the series.

See Website for a Detailed Answer Key

Extra Work Space