2.8 Higher Derivatives

Since the derivative of a function f is itself a function f', we can therefore take the derivative of this function as well $(f')'$. The result is a new function called the **second derivative** of f and is denoted f'' ("f double-prime").

If $y = f(x)$ and using Leibniz notation, then

$$f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Which is abbreviated as

$$f''(x) = \frac{d^2y}{dx^2}$$

Using the D-notation, the symbol D^2 indicates the differentiation operation is to be performed twice. In this way, we have the following notations for the second derivative:

$$y'' = f''(x) = \frac{d^2y}{dx^2} = D^2f(x) = D_x^2f(x)$$

Ex. 1
Find $\frac{d^2y}{dx^2}$ if $y = x^6$.

$y' = 6x^5$ $\quad$ $y'' = \boxed{30x^4}$

Ex. 2
Find the second derivative of $f(x) = 5x^2 + \sqrt{x}$.

$f'(x) = 10x + \frac{1}{2}x^{-\frac{1}{2}}$ $\quad$ $f''(x) = 10 - \frac{1}{4}x^{-3/2} = \boxed{10 - \frac{1}{4x^{3/2}}}$

Ex. 3
Find $f''(1)$ if $f(x) = (2 - x^2)^{10}$. $\quad$ Product Rule and Chain Rule

$f'(x) = 10(2-x^2)^9 \cdot (-2x)$

$= -20x(2-x^2)^9$

$f''(x) = -20x \cdot 9(2-x^2)^8(-2x) + -20(2-x^2)^9$

$= 40x^2 \cdot 9(2-x^2)^8 + -20(2-x^2)^9$

$f''(1) = 40(1)^2 \cdot 9(2-1^2)^8 + -20(2-1)^9 \rightarrow 40 \cdot 9 - 20 = \boxed{340}$

What is the meaning of the second derivative? Since the first derivative is the slope of the tangent line and/or the rate of change of a function, the second derivative is a rate of change of the tangent line slope. This concept will be explored later.

Higher derivatives can also be computed. The **third derivative** is the derivative of the second derivative: $f''' = (f'')'$. Other notations are as follows:

$$y''' = f'''(x) = \frac{d^3x}{dx^3} = D^3f(x) = D_x^3f(x)$$

For derivatives beyond the third derivative the prime notation becomes cumbersome and instead, the fourth derivative for instance is denoted $f^{(4)}$ instead of f''''. In general, the **nth derivative** of f is denoted by $f^{(n)}$ and is obtained by differentiating n times. Written in general terms

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = D^nf(x) = D_x^nf(x)$$

Ex. 4

Find the first five derivatives of $y = x^4 + 2x^3 - 5x^2 + 3x - 6$.

$y' = 4x^3 + 6x^2 - 10x + 3$

$y'' = 12x^2 + 12x - 10$

$y''' = 24x + 12$

$y^{(4)} = 24$

$y^{(5)} = 0$

Ex. 5

If $x^3 + y^3 = 5$, use implicit differentiation to find y''.

$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}5$

$3x^2 + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$

$\frac{dy}{dx} = -\frac{x^2}{y^2}$

$y' = -x^2y^{-2}$ differentiate implicitly, here I use Product instead of Quotient by manipulation $\frac{x^2}{y^2} \to x^2y^{-2}$

$(-1)y'' = x^2(-2y^{-3})\frac{dy}{dx} + 2xy^{-2}$

$(-1)\left[-2x^2y^{-3}\frac{dy}{dx} + 2xy^{-2}\right]$

$\uparrow$ equals $\frac{x^2}{y^2}$

$(-1)\left[-2x^{2}y^{-3}\frac{x^2}{y^2} + 2xy^{-2}\right]$

$(-1)\left[-2x^4y^{-5} + \frac{2x}{y^2}\right]$

$(-1)\left[\frac{-2x^4}{y^5} + \frac{2x}{y^2}\right]$

$\frac{2x^4 - 2xy^3}{y^5} \to \boxed{\frac{2x(x^3 - y^3)}{y^5}}$

Homework Assignment

- Exercise 2.8: #1odd, 2bd, 3 – 6, 7ac

Review Assignments (Does not need to be included as part of your HW Assignment)

- Exercise 2.9: #1 – 3, 4odd, 5 – 14, 15ac, 16, 17