Exercise 2.8 – Practice Problems

1. Find the first and second derivatives of the given functions

a) $f(x) = x^5 - 4x^2 + 1$

$\boxed{f'(x) = 5x^4 - 8x}$

$\boxed{f''(x) = 20x^3 - 8}$

b) $g(x) = 7x^4 + 12x^3 - 4x + 8$

$\boxed{g'(x) = 28x^3 + 36x^2 - 4}$

$\boxed{g''(x) = 84x^2 + 72x}$

c) $f(t) = 2t - \frac{1}{t+1}$ $\rightarrow$ $2t - (t+1)^{-1}$

$\boxed{f'(t) = 2 + \frac{1}{(t+1)^2}}$

$\boxed{f''(t) = \frac{-2}{(t+1)^3}}$

d) $g(t) = \frac{4}{\sqrt{t}}$ $\rightarrow$ $4t^{-\frac{1}{2}}$

$g'(t) = -2t^{-3/2} = \boxed{\frac{-2}{t^{3/2}}}$

$g''(t) = -3t^{-5/2} = \boxed{\frac{-3}{t^{5/2}}}$

e) $y = (2x+1)^8$

$y' = 8(2x+1)^7 \cdot 2 \rightarrow \boxed{16(2x+1)^7}$

$y'' = 16 \cdot 7(2x+1)^6 \cdot 2$

$= \boxed{224(2x+1)^6}$

f) $y = t^3 + \frac{1}{t^3}$ $\rightarrow t^3 + t^{-3}$

$\boxed{y' = 3t^2 - 3t^{-4}}$

$\boxed{y'' = 6t + 12t^{-5}}$

g) $y = \sqrt{x^2+1}$ $\rightarrow (x^2+1)^{\frac{1}{2}}$

$y' = \frac{1}{2(x^2+1)^{\frac{1}{2}}} \cdot 2x \rightarrow \frac{x}{(x^2+1)^{\frac{1}{2}}}$ or $\boxed{x(x^2+1)^{-\frac{1}{2}}}$

$y'' = x\left(-\frac{1}{2(x^2+1)^{3/2}} \cdot 2x\right) + (x^2+1)^{-\frac{1}{2}}(1) = \frac{-x^2}{(x^2+1)^{3/2}} + \frac{1}{(x^2+1)^{\frac{1}{2}}}$

$= \frac{-x^2 + x^2 + 1}{(x^2+1)^{3/2}} \rightarrow \boxed{\frac{1}{(x^2+1)^{3/2}}}$

h) $y = \frac{t}{t-1}$ $\quad t(t-1)^{-1}$

$y' = t(-1(t-1)^{-2}(1)) + 1(t-1)^{-1}$

$= \frac{-t}{(t-1)^{+2}} + \frac{1}{(t-1)} \rightarrow \frac{-t + t - 1}{(t-1)^2} = \boxed{\frac{-1}{(t-1)^2}}$ or $\boxed{-(t-1)^{-2}}$

$y'' = -[-2(t-1)^{-3}(1)]$

$= \boxed{\frac{2}{(t-1)^3}}$

2. Find the third derivative.

a) $f(x) = 1 - 12x + 4x^2 - x^3$

$f'(x) = -12 + 8x - 3x^2$

$f''(x) = 8 - 6x$

$f'''(x) = \boxed{-6}$

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b) $f(x) = \frac{1}{x^5} = x^{-5}$

$f'(x) = -5x^{-6}$

$f''(x) = 30x^{-7}$

$f'''(x) = -210x^{-8} = \boxed{\frac{-210}{x^8}}$

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c) $y = \frac{3}{(4-x)^2} = 3(4-x)^{-2}$

$y' = -6(4-x)^{-3}(-1) = 6(4-x)^{-3}$

$y'' = -18(4-x)^{-4}(-1) = 18(4-x)^{-4}$

$y''' = -72(4-x)^{-5}(-1) = \boxed{\frac{72}{(4-x)^5}}$

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d) $y = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

$y' = \frac{(1+2x)^{-\frac{1}{2}} \cdot \cancel{2}}{\cancel{2}} = (1+2x)^{-\frac{1}{2}}$

$y'' = \frac{-(1+2x)^{-3/2} \cdot \cancel{2}}{\cancel{2}} = -(1+2x)^{-3/2}$

$y''' = \frac{3(1+2x)^{-5/2} \cdot \cancel{2}}{\cancel{2}} = \boxed{\frac{3}{(1+2x)^{5/2}}}$

3. Find the first six derivatives of the function $y = x^5 + x^4 + x^3 + x^2 + x + 1$

$f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$

$f''(x) = 20x^3 + 12x^2 + 6x + 2$

$f'''(x) = 60x^2 + 24x + 6$

$f^4(x) = 120x + 24$

$f^5(x) = 120$

$f^6(x) = 0$

4. If $f(x) = \sqrt{1 + x^3}$, find $f''(2)$.

$f'(x) = \frac{1(1+x^3)^{-\frac{1}{2}}}{2} \cdot 3x^2 = \frac{3x^2}{2(1+x^3)^{\frac{1}{2}}}$ or $\frac{3x^2(1+x^3)^{-\frac{1}{2}}}{2}$

$f''(x) = \frac{3}{2}\left[x^2 \cdot -\frac{1}{2(1+x^3)^{3/2}} \cdot 3x^2 + 2x(1+x^3)^{-\frac{1}{2}}\right]$ at $x=2$

$= \frac{3}{2}\left[2^2 \cdot \frac{-3(2)^2}{2(1+2^3)^{3/2}} + 2(2)(1+2^3)^{-\frac{1}{2}}\right] \rightarrow \frac{3}{2}\left[-\frac{48}{54} + \frac{4}{3}\right] = \boxed{\frac{2}{3}}$

5. If $g(x) = \frac{1}{\sqrt{3x+4}}$, find $g'''(4)$.

$g(x) = (3x+4)^{-\frac{1}{2}}$

$g'(x) = -\frac{1(3x+4)^{-\frac{3}{2}}}{2} \cdot 3 = -\frac{3}{2}(3x+4)^{-\frac{3}{2}}$

$g''(x) = \frac{9}{4}(3x+4)^{-5/2} \cdot 3 = \frac{27}{4}(3x+4)^{-\frac{5}{2}}$

$g'''(x) = -\frac{135}{8}(3x+4)^{-\frac{7}{2}} \cdot 3 = \frac{-405}{8(3x+4)^{7/2}}$ at $x=4$

$\frac{-405}{8(16)^{7/2}} = \frac{-405}{8(4)^7}$

$\boxed{\frac{-405}{131072}}$

6. If $f(x) = x^n$, find $f^{(n)}(x)$

$f'(x) = nx^{n-1}$

$f''(x) = (n-1)nx^{n-2}$

$f'''(x) = (n-2)(n-1)nx^{n-3}$

and so on.

This is what we can a factorial

$\boxed{n!}$ means

$n(n-1)(n-2)(n-3)\ldots$

7. Find y'' by implicit differentiation.

a) $x^4 + y^4 = 1$ *

$\frac{d}{dx}(x^4+y^4) = \frac{d}{dx} 1 \rightarrow 4x^3 + 4y^3\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$

$y'' = -\left[\frac{y^3(3x^2) - [x^3(3y^2)\frac{dy}{dx}]}{(y^3)^2}\right]$ $\frac{dy}{dx} = -\frac{x^3}{y^3}$ $\rightarrow -\left[\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}\right] \rightarrow -\left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$

* but $\boxed{(x^4+y^4)=1}$ $\quad -\frac{3x^2(y^4+x^4)}{y^7} = \boxed{\frac{-3x^2}{y^7}}$

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b) $x^2 - y^2 = 1$

$\frac{d}{dx}(x^2-y^2) = \frac{d}{dx} 1 \rightarrow 2x - 2y\frac{dx}{dx} = 0$ $\boxed{\frac{dy}{dx} = \frac{x}{y}}$

$y'' = \frac{y - x\frac{dy}{dx}}{y^2} \rightarrow \frac{y - x\left(\frac{x}{y}\right)}{y^2} \rightarrow \frac{\frac{y^2-x^2}{y}}{y^2} \rightarrow \frac{y^2-x^2}{y^3} = \frac{-1(x^2-y^2)}{y^3}$ $= 1$

$\boxed{\frac{-1}{y^3}}$

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c) $x^3 + y^3 = 6xy$ $\quad \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}6xy \rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 6\left[x\frac{dy}{dx} + y\right]$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = -3x^2 + 6y \rightarrow \frac{dy}{dx} = \frac{-3x^2+6y}{3y^2-6x} = \frac{-x^2+2y}{y^2-2x} = \frac{x^2-2y}{2x-y^2}$

* This takes a long time

see detailed answer key from original book.

$\boxed{\frac{16}{(2x-y^2)^3}}$

8. Find a quadratic function f such that $f(3) = 33$, $f'(3) = 22$, and $f''(3) = 8$

Let $ax^2 + bx + c = 33$ then $2ax + b = 22$ and $2a = 8$

this gives $a = 4$ $\longrightarrow$ $2(4)x + b = 22$ when $x = 3$

$2(4)(3) + b = 22$

$4x^2 + (-2)x + c = 33$ when $x = 3$ $\quad b = -2$

$4(3)^2 - 2(3) + c = 33$

$36 - 6 + c = 33$ $\quad c = -3$

$30 + c = 33$

therefore : $\boxed{4x^2 - 2x - 3 = f(x)}$

9. Suppose that $f(x) = g(x)h(x)$.

a) Express f'' in terms of g, g', g'', h, h', h''

$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f''(x) = g(x)h''(x) + h'(x)g'(x) + h(x)g''(x) + g'(x)h'(x)$

$= g''(x)h(x) + h''(x)g(x) + 2g'(x)h'(x)$

b) Find a similar expression for f'''

From above solution

$g''(x)h'(x) + h(x)g'''(x) + h'(x)g'(x) + g(x)h'''(x) + 2[g'(x)h''(x) + g''(x)h'(x)]$

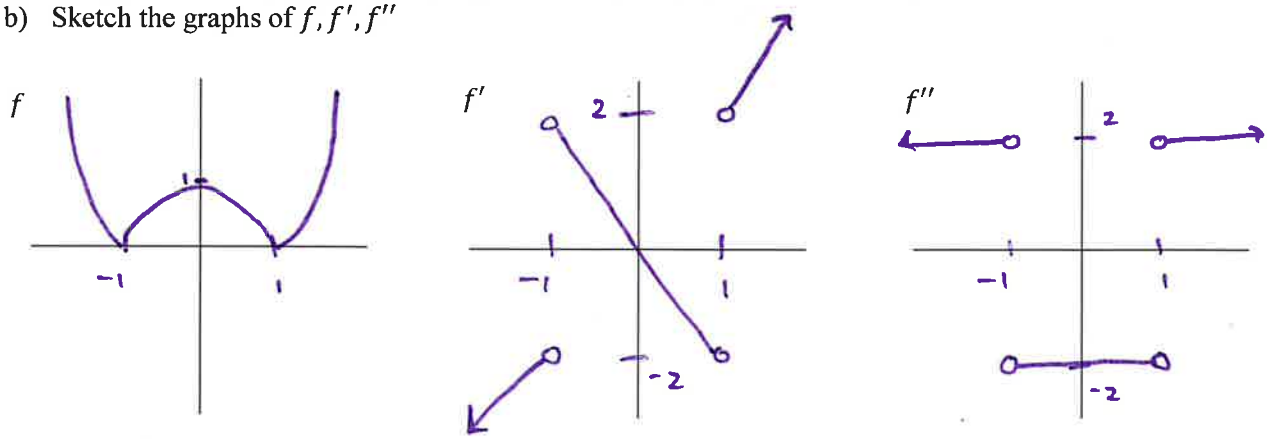
$3g''(x)h'(x) + h(x)g'''(x) + 3h''(x)g'(x) + g(x)h'''(x)$

$g'''(x)h(x) + h'''(x)g(x) + 3g''(x)h'(x) + 3h''(x)g'(x)$

10.

a) If $f(x) = |x^2 - 1|$, find f' and f'' and state their domains.

b) Sketch the graphs of f, f', f''



$f(x) = |x^2 - 1| = x^2 - 1$ if $|x| \geq 1$

$-x^2 + 1$ if $|x| < 1$

we see sharp corners so $f'(x)$ does not exist at ± 1

$f'(x) = 2x$ if $|x| > 1$

$-2x$ if $|x| < 1$

$f''(x) = 2$ if $|x| > 1$

-2 if $|x| < 1$