2.7 Implicit Differentiation

The only kinds of functions that we have learned how to differentiate are ones that are explicitly defined, i.e., expressing one variable in terms of another, such as

$$y = x^2 \qquad \text{or} \qquad y = \frac{\sqrt{x^3+5}}{6x-1}$$

What about functions that are defined implicitly, like that of a circle?

$$x^2 + y^2 = 25$$

One way is to solve the equation for y which gives $y = \pm\sqrt{25-x^2}$ and then differentiating two functions:

$$f(x) = \sqrt{25-x^2} \qquad \text{and} \qquad g(x) = -\sqrt{25-x^2}$$

This method is cumbersome, and a better approach is a method called **implicit differentiation**.

Ex. 1

derive with respect to x and y then isolate $\frac{dy}{dx}$

(a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

(b) Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-4, 3)$.

a) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx} 25$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$

b) $\frac{dy}{dx} = -\frac{(-4)}{3}$

$= \frac{4}{3}$

$y = \frac{4}{3}x + b$

$3 = \frac{4}{3}(-4) + b$

$3 = \frac{-16}{3} + b$

$\frac{9}{3} + \frac{16}{3} = b$

$b = \frac{25}{3}$

$\boxed{y = \frac{4}{3}x + \frac{25}{3}}$

Ex. 2

(a) Find $\frac{dy}{dx}$ if $2x^5 + x^4y + y^5 = 36$.

(b) Find the slope of the tangent to the curve $2x^5 + x^4y + y^5 = 36$ at the point $(1, 2)$.

a) $\frac{d}{dx}(2x^5 + x^4y + y^5) = \frac{d}{dx} 36$

$10x^4 + x^4\frac{dy}{dx} + y(4x^3) + 5y^4\frac{dy}{dx} = 0$

$10x^4 + 4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$

$x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = -10x^4 - 4x^3y$

$\frac{dy}{dx}(x^4 + 5y^4) = -10x^4 - 4x^3y$

$\frac{dy}{dx} = -\frac{10x^4 + 4x^3y}{x^4 + 5y^4}$

b) Evaluate at $(1, 2)$

$\frac{dy}{dx} = -\frac{10(1)^4 + 4(1)^3(2)}{1^4 + 5(2)^4}$

$= -\frac{18}{81} = \boxed{-\frac{2}{9}}$

Ex. 3

Find y' if $x^2 + \sqrt{y} = x^2y^3 + 5$.

$\frac{d}{dx}(x^2 + y^{\frac{1}{2}}) = \frac{d}{dx}(x^2y^3 + 5)$

$2x + \frac{1}{2y^{\frac{1}{2}}}\frac{dy}{dx} = x^2(3y^2)\frac{dy}{dx} + 2xy^3 + 0$

$2x + \frac{1}{2y^{\frac{1}{2}}}\frac{dy}{dx} = 3x^2y^2\frac{dy}{dx} + 2xy^3$

$\frac{1}{2y^{\frac{1}{2}}}\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} = 2xy^3 - 2x$

$\frac{dy}{dx}\left(\frac{1}{2y^{\frac{1}{2}}} - 3x^2y^2\right) = 2xy^3 - 2x$

$\frac{dy}{dx} = \frac{2xy^3 - 2x}{\frac{1}{2\sqrt{y}} - 3x^2y^2}$

$= \frac{\cancel{-}2x(1 - y^3)}{\cancel{-1}(3x^2y^2 - \frac{1}{2\sqrt{y}})}$

$\boxed{y' = \frac{2x(1 - y^3)}{3x^2y^2 - \frac{1}{2\sqrt{y}}}}$

Homework Assignment

- Exercise 2.7: #1 – 3odd, 4abc (try to sketch it if you like), 5a (try to sketch it if you like), 6, 7