**Exercise 2.7 – Practice Problems**

1. Use Implicit differentiation to find $\frac{dy}{dx}$. Follow this process

a) $x^2 - y^2 = 1$

$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx} 1$

$2x - 2y\frac{dy}{dx} = 0$

$-2y\frac{dy}{dx} = -2x$

$\boxed{\frac{dy}{dx} = \frac{x}{y}}$

b) $x^3 + y^3 = 6$

$3x^2 + 3y^2\frac{dy}{dx} = 0$

$3y^2\frac{dy}{dx} = -3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$

$\boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$

c) $xy = 4$ Product Rule

$x(1)\frac{dy}{dx} + 1y = 0$

$\frac{dy}{dx}x + y = 0$

$\frac{dy}{dx}x = -y$

$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$

d) $x^2 + xy + y^2 = 1$ Product Rule

$2x + x(1)\frac{dy}{dx} + 1y + 2y\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y - 2x$

$\frac{dy}{dx}(x+2y) = -(y+2x)$

$\frac{dy}{dx} = -\frac{(y+2x)}{(x+2y)}$

e) $x^3 + y^3 = 6xy$

$3x^2 + 3y^2\frac{dy}{dx} = 6[x\frac{dy}{dx} + y]$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = -3x^2 + 6y$

$\frac{dy}{dx}(3y^2 - 6x) = -3x^2 - 6y \rightarrow \frac{dy}{dx} = -\frac{(3x^2-6y)}{(3y^2-6x)}$

$\boxed{\frac{dy}{dx} = -\frac{(x^2-2y)}{(y^2-2x)}}$

f) $2xy^2 - y^3 = x^2$

$2[x2y\frac{dy}{dx} + 2y^2] - 3y^2\frac{dy}{dx} = 2x$

$2x4y\frac{dy}{dx} + 4y^2 - 3y^2\frac{dy}{dx} = 2x$

$\frac{dy}{dx}(8xy - 3y^2) = 2x - 4y^2$

$\boxed{\frac{dy}{dx} = \frac{2x-4y^2}{8xy-3y^2}}$

g) $\sqrt{x} + \sqrt{y} = 1$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\frac{dy}{dx}\frac{1}{2\sqrt{y}} = -\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}}$

$\boxed{\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}}$

h) $\frac{2x}{x+y} = y \rightarrow 2x = y(x+y) = 2x = xy + y^2$

$2 = x\frac{dy}{dx} + y + 2y\frac{dy}{dx}$

$2 - y = \frac{dy}{dx}(x+2y)$

$\boxed{\frac{2-y}{x+2y} = \frac{dy}{dx}}$

2. Find the slope of the tangent line to the curve at the given point.

a) $x^2 + 4y^2 = 5; (1, -1)$

$2x + 8y\frac{dx}{dx} = 0$ $\qquad \frac{dy}{dx}$ at $(1,-1)$

$\frac{dy}{dx} = \frac{-2x}{8y}$ $\qquad = \frac{-1}{4(-1)}$

$\frac{dy}{dx} = -\frac{x}{4y}$ $\qquad = \boxed{\frac{1}{4}}$

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b) $x^4 + y^4 = 17; (2, 1)$

$4x^3 + 4y^3\frac{dy}{dx} = 0$ $\qquad -\frac{2^3}{1^3} = \boxed{-8}$

$\frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$ at $(2,1)$

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c) $x^2 + x^3y^2 - y^3 = 13; (1, -2)$

$2x + x^3 2y\frac{dy}{dx} + 3x^2y^2 - 3y^2\frac{dy}{dx} = 0$

$x^3 2y\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x - 3x^2y^2$

$\frac{dy}{dx}(x^3 2y - 3y^2) = -(2x + 3x^2y^2)$

$\frac{dy}{dx} = -\frac{(2x + 3x^2y^2)}{(x^3 2y - 3y^2)}$ at $(1,-2)$

$\frac{dy}{dx} = -\frac{(2(1) + 3(1)^2(-2)^2)}{((1^3)2(-2) - 3(-2)^2)} = \frac{2+12}{-16}$

$-\frac{14}{-16} = \boxed{\frac{7}{8}}$

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d) $y^2 = 2xy - 3; (2, 3)$

$2y\frac{dy}{dx} = 2\left[x\frac{dy}{dx} + y\right]$

$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y$

$\frac{dy}{dx}(-2x + 2y) = 2y$

$\frac{dy}{dx} = \frac{2y}{-2x+2y}$ at $(2,3)$

$\frac{dy}{dx} = \frac{2(3)}{-2(2)+2(3)} = \frac{6}{2} = \boxed{3}$

e) $\sqrt{x+y}+\sqrt{xy}=4;(2,2)$

$\frac{1}{2\sqrt{(x+y)}}\left(1+\frac{dy}{dx}\right)+\frac{1}{2\sqrt{xy}}\cdot\left(x\frac{dy}{dx}+y\right)=0$

$\frac{1}{2\sqrt{x+y}}+\frac{1}{2\sqrt{x+y}}\cdot\frac{dy}{dx}+\frac{1}{2\sqrt{xy}}\cdot\frac{x\,dy}{dx}+\frac{y}{2\sqrt{xy}}=0$

$\frac{dy}{dx}\left[\frac{1}{2\sqrt{x+y}}+\frac{x}{2\sqrt{xy}}\right]=-\frac{y}{2\sqrt{xy}}-\frac{1}{2\sqrt{x+y}}$

$\frac{-\frac{y}{2\sqrt{xy}}-\frac{1}{2\sqrt{x+y}}}{\frac{1}{2\sqrt{x+y}}+\frac{x}{2\sqrt{xy}}}=\frac{dy}{dx}$ at $(2,2)$

$\frac{-\frac{2}{4}-\frac{1}{4}}{\frac{1}{4}+\frac{2}{4}}\rightarrow\frac{-\frac{3}{4}}{\frac{3}{4}}=\boxed{-1}$

f) $\frac{1}{x}+\frac{1}{y}=1;\left(\frac{3}{2},3\right)$ $x^{-1}+y^{-1}=1$

$-1x^{-2}+-1y^{-2}\frac{dy}{dx}=0$

$-\frac{1}{y^2}\frac{dy}{dx}=\frac{1}{x^2}$

$\frac{dy}{dx}=-\frac{y^2}{x^2}$ at $(\frac{3}{2},3)$

$-\frac{(3)^2}{\left(\frac{3}{2}\right)^2}=-\frac{9}{\frac{9}{4}}=\boxed{-4}$

3. Find the equation to the tangent line to the curve at the given point.

a) $2x^2-y^2=1;(-1,-1)$

$4x-2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-4x}{-2y}=\frac{2x}{y}$

$\frac{2(-1)}{-1}$

$\frac{dy}{dx}=2$

$y=2x+b$

$-1=-2+b$

$1=b$

$\boxed{y=2x+1}$

b) $x^3+y^3=9;(2,1)$

$3x^2+3y^2\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-3x^2}{3y^2}$

$=-\frac{x^2}{y}$ at $x=2$, $y=1$

$\frac{dy}{dx}=-\frac{4}{1}=-4$

$y=-4x+b$

$1=-4(2)+b$

$9=b$

$\boxed{y=-4x+9}$

c) $y^5 + x^2y^3 = 10; (-3, 1)$

$$5y^4\frac{dy}{dx} + x^2(3y^2)\frac{dy}{dx} + 2xy^3 = 0$$

$$5y^4\frac{dy}{dx} + 3x^2y^2\frac{dy}{dx} = -2xy^3$$

$$\frac{dy}{dx}(5y^4+3x^2y^2) = -2xy^3$$

$$\frac{dy}{dx} = \frac{-2xy^3}{(5y^4+3x^2y^2)} \quad \text{at } (-3,1)$$

$$\frac{dy}{dx} = \frac{-2(-3)(1)^3}{(5(1)^4+3(-3)^2(1)^2}$$

$$= \frac{6}{5+27} = \frac{6}{32} = \boxed{\frac{3}{16}}$$

$$y = \frac{6}{23}x + b$$

$$1 = -\frac{18}{23} + b \qquad b = \frac{25}{16}$$

$$\boxed{y = \frac{3}{16}x + \frac{25}{16}}$$

d) $(x + y)^3 = x^3 + y^3; (-1, 1)$

$$3(x+y)^2\cdot\left(1+\frac{dy}{dx}\right) = 3x^2+3y^2\frac{dy}{dx}$$

$$3(x+y)^2 + \frac{dy}{dx}(3(x+y)^2) = 3x^2+3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[3(x+y)^2 - 3y^2\right] = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx} = \frac{3x^2-3(x+y)^2}{3(x+y)^2-3y^2} \quad \text{at } (-1,1)$$

$$\frac{3(-1)^2-3(0)^2}{3(0)^2-3(1)^2} = \frac{3}{-3} = \boxed{-1}$$

$$y = -x + b$$

$$1 = 1 = b$$

$$0 = b$$

$$\boxed{y = -x}$$

4.

a) Use implicit differentiation to find the slope of the tangent line to the ellipse $9x^2 + 4y^2 = 36$ at the point $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$

$$18x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-18x}{8y}$$

$$\frac{dy}{dx} = -\frac{9x}{4y} \quad \text{at } \left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$$

$$= \frac{-9\sqrt{2}}{4\left(\frac{3}{2}\sqrt{2}\right)} = \frac{-9\sqrt{2}}{6\sqrt{2}} = \boxed{-\frac{3}{2}}$$

only this one because $y > 0$

b) Find the slope in part (a) by first solving for $y$ explicitly as a function of $x$.

$$9x^2+4y^2 = 36 \rightarrow 4y^2 = -9x^2+36 \rightarrow y^2 = -\frac{9}{4}x^2+9 \qquad y_1 = \sqrt{-\frac{9}{4}x^2+9}$$

$$y = 3\sqrt{-\frac{x^2}{4}+1}$$

$$y' \frac{3}{2\sqrt{-\frac{x^2}{4}+1}}\cdot\frac{-1x}{2} \rightarrow \frac{-3x}{4\sqrt{\left(\frac{-x^2+4}{4}\right)}} \quad \text{at } x = \sqrt{2} \quad y' = \frac{-3\sqrt{2}}{4\sqrt{\frac{1}{2}}} \times \frac{-3\sqrt{2}\sqrt{2}}{4} = \boxed{\frac{-3}{2}}$$

c) Find the equation of the tangent line.

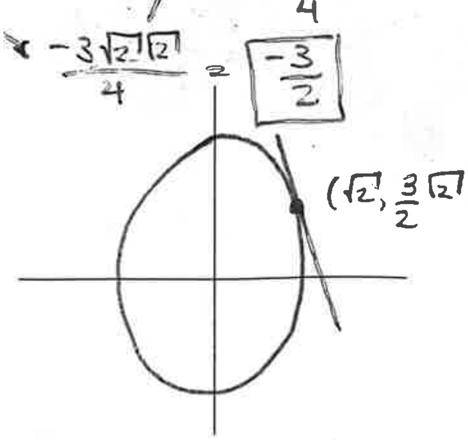
$$y = -\frac{3}{2}x + b$$

$$\frac{3}{2}\sqrt{2} = -\frac{3}{2}\sqrt{2} + b$$

$$\frac{3}{\sqrt{2}} = -\frac{3}{\sqrt{2}} + b \qquad b = \frac{6}{\sqrt{2}}$$

$$\boxed{y = -\frac{3}{2}x + \frac{6}{\sqrt{2}}}$$

d) Sketch the ellipse and the tangent line.



$\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$

USE DESMOS TO HELP

5.

a) Find an equation of the tangent line to the circle $x^2 + y^2 + 2x - 4y - 20 = 0$ at the point $(2, -2)$.

$$2x + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 4\frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx}(2y-4) = -2(x-1)$$

$$\frac{dy}{dx} = \frac{-2(x-1)}{2(y-2)} = -\frac{(x-1)}{(y-2)}$$

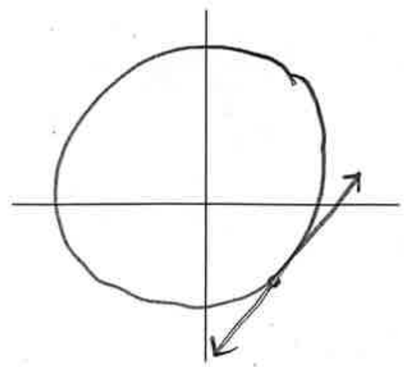
at $(2,-2)$ $\quad \frac{dy}{dx} = -\frac{(2-1)}{(-2-2)} = -\frac{1}{-4} = \boxed{\frac{1}{4}}$

$$y = \frac{1}{4}x + b$$

$$-2 = \frac{1}{2} + b$$

$$-\frac{3}{2} = b$$

$$\boxed{y = \frac{1}{4}x - \frac{3}{2}}$$



b) Sketch the circle and the tangent line.

USE DESMOS AND DUPLICATE

6. The curve with the equation $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ is called a *lemniscate* and is shown in the figure below.

a) Find $y'$

$$2[2(x^2+y^2)\cdot(2x+2y\frac{dy}{dx})] = 25(2x - 2y\frac{dy}{dx})$$

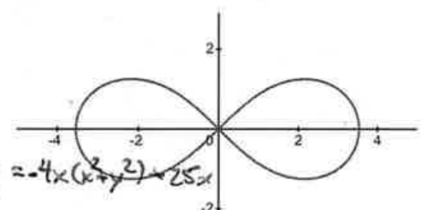
$$8(x^2+y^2)(x+y\frac{dy}{dx}) = 25\cdot 2(x-y\frac{dy}{dx})$$

$$4(x^2+y^2)(x+y\frac{dy}{dx}) = 25(x-y\frac{dy}{dx})$$

$$4x(x^2+y^2) + 4y\frac{dy}{dx}(x^2+y^2) = 25x - 25y\frac{dy}{dx}$$

$$4y\frac{dy}{dx}(x^2+y^2) + 25y\frac{dy}{dx} = -4x(x^2+y^2) + 25x$$

$$\frac{dy}{dx} = \frac{-4x(x^2+y^2) + 25x}{4y(x^2+y^2) + 25y} \rightarrow \frac{dy}{dx} = \left[\frac{x(25-4(x^2+y^2))}{y(25+4(x^2+y^2))}\right]$$




b) Find the equation of the tangent line to the lemniscate at the point $(-3, 1)$.

$\frac{dy}{dx}$ at $x=-3$, $y=1$

$$\left[\frac{-3(25-4((-3)^2+1^2))}{1(25+4(-3)^2+1^2)}\right] \rightarrow \frac{-3(25-40)}{1(25+40)} = \frac{+45}{65} = \boxed{\frac{9}{13}}$$

$$y = \frac{9}{13}x + b \rightarrow 1 = \frac{9}{13}(-3) + b \qquad b = \frac{40}{13}$$

$$\boxed{y = \frac{9}{13}x + \frac{40}{13}}$$

c) Find the points on the lemniscate where the tangent line is horizontal.

when is $\frac{dy}{dx} = 0$ so consider numerator $x(25-4(x^2+y^2)) = 0$

so $x = 0$ but that gives $y = 0$ so no tangent. Can't divide by 0.

$$25 - 4(x^2+y^2) = 0 \rightarrow 4x^2 + 4y^2 = 25$$

$$25 - 4x^2 - 4y^2 = 0$$

$$\boxed{x^2 + y^2 = \frac{25}{4}} \rightarrow \boxed{x^2 =} \frac{25}{4} - y^2 = \boxed{\frac{25-4y^2}{4}}$$

From original:

$$2\left(\frac{25}{4}\right)^2 = 25\left(\frac{25-4y^2}{4} - y^2\right)$$

$$2\left(\frac{625}{16}\right) = 25\left(\frac{25-4y^2-4y^2}{4}\right)$$

$$\frac{625}{8} = 25\left(\frac{25-8y^2}{4}\right)$$

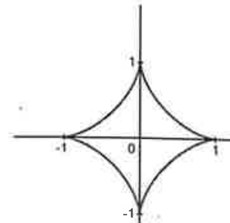
$$\frac{25}{8} = \frac{25-8y^2}{4} \rightarrow \frac{25}{2} = 25 - 8y^2$$

$$y^2 = \frac{25}{16} \qquad \boxed{y = \pm\frac{5}{4}}$$

$$\boxed{x = \pm\frac{5\sqrt{3}}{4}}$$

$$\sqrt[3]{\frac{\frac{3\sqrt{3}}{8}}{\frac{1}{8}}} \downarrow \sqrt[3]{\sqrt{3}^3} \downarrow \sqrt{3}$$

7. The curve with the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is called an *asteroid* and is shown in the figure below.



a) Find $y'$ $\quad \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

$$\frac{2}{3y^{\frac{1}{3}}}\frac{dy}{dx} = -\frac{2}{3x^{\frac{1}{3}}} \rightarrow \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\sqrt[3]{\frac{y}{x}}$$

b) Find the equation of the tangent line to the asteroid at the point $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$.

$$y = -\sqrt[3]{\frac{y}{x}}\,x + b \rightarrow \frac{3\sqrt{3}}{8} = -\sqrt[3]{\frac{\frac{3\sqrt{3}}{8}}{\frac{1}{8}}}\left(\frac{1}{8}\right) + b$$

$$\frac{3\sqrt{3}}{8} = -\frac{\sqrt[3]{3\sqrt{3}}}{8} + b \rightarrow \frac{3\sqrt{3}}{8} = -\frac{\sqrt[3]{\sqrt{3}^3}}{8} + b \rightarrow \frac{3\sqrt{3}}{8} = -\frac{\sqrt{3}}{8} + b$$

$$b = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\boxed{y = \sqrt{3}x + \frac{\sqrt{3}}{2}}$$

c) Find the points on the asteroid where the tangent line has slope 1.

$$-\sqrt[3]{\frac{y}{x}} = 1 \quad -\frac{y}{x} = 1 \quad -y = x \rightarrow y = -x$$

sub into original

$$x^{2/3} + (-x)^{2/3} = 1$$
$$x^{2/3} + x^{2/3} = 1 \rightarrow 2x^{2/3} = 1 \rightarrow x^{2/3} = \frac{1}{2} \rightarrow x^2 = \frac{1}{8} \rightarrow x = \pm\frac{1}{2\sqrt{2}}$$

if $\boxed{x = \pm\frac{1}{2\sqrt{2}} \quad y = \pm\frac{1}{2\sqrt{2}}}$

since $x = -y$

points of intersect

8. Use implicit differentiation to show that an equation of the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

At the point $(x_0, y_0)$ is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

Point slope: $y - y_1 = m(x - x_1)$

this case

$(x_1, y_1) \rightarrow (x_0, y_0)$

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$2xb^2 + 2ya^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xb^3}{2ya^2}$$

$$\frac{dy}{dx} = \frac{-xb^2}{ya^2}$$

$$y - y_0 = \left(\frac{-xb^2}{ya^2}\right)(x - x_0)$$

$$ya^2(y - y_0) = -xb^2(x - x_0)$$

$$\frac{y^2a^2 - ya^2y_0}{b^2} = \frac{-x^2b^2 + xb^2x_0}{b^2}$$

$$\frac{y^2a^2 - ya^2y_0}{b^2a^2} = \frac{xx_0 - x^2}{a^2}$$

$$\frac{y^2}{b^2} - \frac{yy_0}{b^2} = \frac{xx_0}{a^2} - \frac{x^2}{a^2}$$

$$\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2}}_{1} = \frac{x_0x}{a^2} + \frac{y_0y}{b^2}$$

$$\boxed{1 = \frac{x_0x}{a^2} + \frac{y_0y}{b^2}}$$

Helps to consider $f(x) = y$

9. Suppose $f$ is a function such that $x[f(x)]^3 + x^2 f(x) = 3$ and $f(2) = 1$. Find $f'(2)$.

$$xy^3 + x^2y = 3$$

$$x(3y^2)\frac{dy}{dx} + y^3 + x^2\frac{dy}{dx} + 2xy = 0$$

$$3xy^2\frac{dy}{dx} + x^2\frac{dy}{dx} = -2xy - y^3$$

$$\frac{dy}{dx} = -\frac{(2xy + y^3)}{3xy^2 + x^2)}$$

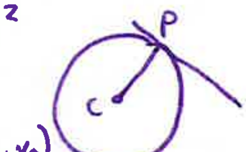
sub back $y = f(x)$

$$f'(2) = -\frac{2(2)f(2) + [f(2)]^3}{3(2)(f(2))^2 + 2^2}$$

$$= -\frac{4(1) + (1)^3}{3(2)(1)^2 + 4} = -\frac{5}{10} \quad \boxed{-\frac{1}{2}}$$

10. Use implicit differentiation to show that any tangent line at a point $P$ to a circle with a centre $C$ is perpendicular to the radius $CP$.

Let the equation of the circle by $(x - x_0)^2 + (y - y_0)^2 = r^2$




put center at $(x_0, y_0)$

point on the tangent is $(x, y)$

$$\frac{d}{dx}\left[(x - x_0)^2 - (y - y_0)^2\right] = r^2$$

$$2(x - x_0) - 2(y - y_0)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(x - x_0)}{(y - y_0)}$$

Negative Reciprocals.

m of CP is $\frac{y - y_0}{x - x_0}$

11. Use implicit differentiation to show that, whenever a hyperbola with equation $x^2 - y^2 = k$ intersects a hyperbola with equation $xy = c$, the tangent lines at the points of intersection are perpendicular.

Intersection occurs at an infinite number of points

Compute derivatives in the abstract

$$\frac{d}{dx}\left[x^2 - y^2 = k\right]$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{x}{y}}$$

$$\frac{d}{dx}\left[xy = c\right]$$

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Since derivatives are negative reciprocals slopes will be perpendicular at all points