This comes back to composition of functions

2.6 The Chain Rule

How do we compute the derivative of a function such as $F(x) = \sqrt{2x^2+3}$? This type of function is a composite function meaning it can be built up from simpler functions.

$$y = f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = 2x^2 + 3$$

Then

$$f(g(x)) = f(2x^2+3) = \sqrt{2x^2+3} = F(x)$$

That is, $F = f \circ g$. We can compute the derivative using the Chain Rule which will give the derivative of F in terms of the derivatives of f and g. Interpreting the derivative as a rate of change the $\frac{du}{dx}$ would represent the rate of change of u with respect to x, $\frac{dy}{du}$ as the rate of change of y with respect to u, and $\frac{dy}{dx}$ is the rate of change of y with respect to x.

If u changes twice as fast as x ($\frac{du}{dx} = 2$) and y changes three times as fast as u ($\frac{dy}{du} = 3$), then you would expect y to change six times as fast as x. Looking at it algebraically it would look like this.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

If the derivatives of $g'(x)$ and $f'(x)$ both exist and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists and is given by the product $F'(x) = f'(g(x))g'(x)$; that is

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

To use the Chain Rule, you differentiate a function from the *inside* to the *outside*.

The Chain Rule in Leibniz Notation

If $y = f(u)$, where $u = g(x)$, and f and g are differentiable, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Ex. 1

Find $F'(x)$ if $F(x) = \sqrt{2x^2+3}$.

let $g(x) = 2x^2+3$

$f(x) = \sqrt{g(x)}$

$F'(x) = f'(g(x))\,g'(x)$

$= \frac{1}{2}(2x^2+3)^{-\frac{1}{2}} \cdot 4x$

$= \frac{4x}{2(2x^2+3)^{\frac{1}{2}}}$

$\frac{2x}{\sqrt{2x^2+3}}$

$\boxed{F'(x) = \frac{2x}{\sqrt{2x^2+3}}}$

Ex. 2

If $y = u^{10} + u^5 + 2$, where $u = 1 - 3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{du} = 10u^9 + 5u^4$

$\frac{du}{dx} = -6x$

$\frac{dy}{dx} = 10u^9 + 5u^4 (-6x)$

$= (10(-2)^9 + 5(-2)^4)(-6(1))$

$= (10(-512) + 5(16))(-6)$

$\left.\frac{dy}{dx}\right|_{x=1} = (-5040)(-6) = \boxed{30\,240}$

$u = 1 - 3x^2$

u when $x = 1$

$u = 1 - 3(1)^2$

$\boxed{u = -2}$

When the outer function is a power function a special case of the Chain Rule occurs. Suppose that $y = f(u) = u^n$, where $u = g(x)$. Using the Power Rule and then the Chain Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

Power Rule Combined with Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Or

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Ex. 3

If $y = (x^2 - x + 2)^8$, find $\frac{dy}{dx}$.

$y' = 8(x^2 - x + 2)^7 \cdot (2x - 1)$

• Take derivative of the 'outside' than don't forget to take the derivative of the inside

Power and Chain Rule!

Ex. 4

Find $f'(x)$ for the following function.

$f(x) = \frac{1}{\sqrt[3]{1-x^4}}$ consider writing $\rightarrow \frac{1}{(1-x^4)^{\frac{1}{3}}} \rightarrow (1-x^4)^{-\frac{1}{3}}$

$$f'(x) = -\frac{1}{3}(1-x^4)^{-4/3}(-4x^3)$$

$$f'(x) = \frac{4x^3}{3(1-x^4)^{4/3}} \rightarrow \boxed{\frac{4x^3}{3\sqrt[3]{1-x^4}^{\,4}}}$$

Ex. 5

Differentiate $s = \left(\frac{2t-1}{t+2}\right)^6$. Need to use the quotient rule for the inside

$$s' = 6\left(\frac{2t-1}{t+2}\right)^5 \cdot \frac{(t+2)(2)-(2t-1)(1)}{(t+2)^2}$$

$$= 6\left(\frac{2t-1}{t+2}\right)^5 \cdot \frac{2t+4-2t+1}{(t+2)^2}$$

$$= 6\left[\frac{(2t-1)^5}{(t+2)^5}\right] \cdot \frac{5}{(t+2)^2}$$

$$= \frac{6(2t-1)^5}{(t+2)^5} \cdot \frac{5}{(t+2)^2} \rightarrow \boxed{s' = \frac{30(2t-1)^5}{(t+2)^7}}$$

Ex. 6

Find the derivative of the function $f(x) = (x^2+1)^3(2-3x)^4$.

Product Rule with Chain Rule

$$f'(x) = (x^2+1)^3(4)(2-3x)^3(-3) + (2-3x)^4(3)(x^2+1)^2(2x)$$

$$= -12(x^2+1)^3(2-3x)^3 + 6x(2-3x)^4(x^2+1)^2$$

Factor out common factor $-6(x^2+1)^2(2-3x)^3$

$$= -6(x^2+1)^2(2-3x)^3[2(x^2+1) - x(2-3x)]$$

$$= -6(x^2+1)^2(2-3x)^3[2x^2+2-2x+3x^2]$$

$$\boxed{f'(x) = -6(x^2+1)^2(2-3x)^3(5x^2-2x+2)}$$

Ex. 7

If h is a differentiable function find the derivatives of the following functions.

(a) $F(x) = [h(x)]^3$

$$F'(x) = 3[h(x)]^2 \cdot h'(x)$$

(b) $G(x) = h(x^3)$

$$G'(x) = h'(x^3) \cdot 3x^2$$

$$= 3x^2 h'(x^3)$$

Ex. 8

Find y' if $y = \sqrt{x + \sqrt{x^2+1}}$.

$$y = (x + (x^2+1)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x + (x^2+1)^{\frac{1}{2}})^{\frac{1}{2}-1} \cdot \left(1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right)$$

$$y' = \frac{1}{2}(x + (x^2+1)^{\frac{1}{2}})^{-\frac{1}{2}}\left(1 + \frac{x}{(x^2+1)^{\frac{1}{2}}}\right)$$

$$y' = \frac{1}{2\sqrt{x+\sqrt{x^2+1}}} \cdot \left[1 + \frac{x}{\sqrt{x^2+1}}\right]$$