Exercise 2.6 – Practice Problems

1. Find the derivatives of the following functions

a) $F(x) = (5-3x)^7$

$F'(x) = 7(5-3x)^6 \cdot (-3)$

$= \boxed{-21(5-3x)^6}$

b) $F(x) = (2x^2+1)^{20}$

$F'(x) = 20(2x^2+1)^{19} \cdot 4x$

$= \boxed{80x(2x^2+1)^{19}}$

c) $G(x) = (x^3+x^2-2)^{\frac{3}{4}}$

$G'(x) = \frac{3}{4}(x^3+x^2-2)^{-\frac{1}{4}}(3x^2+2x)$

$= \dfrac{3(3x^2+2x)}{4\sqrt[4]{x^3+x^2-2}} = \boxed{\dfrac{9x^2+6x}{4\sqrt[4]{x^3+x^2-2}}}$

d) $G(x) = \sqrt{x^4-x+1}$

$G'(x) = \dfrac{1(x^4-x+1)^{-\frac{1}{2}}}{2} \cdot (4x^3-1)$

$= \boxed{\dfrac{4x^3-1}{2(\sqrt{x^4-x+1})}}$

e) $y = \sqrt[4]{x^2+x}$

$y = (x^2+x)^{\frac{1}{4}}$

$y' = \dfrac{1(x^2+x)^{-3/4}}{4} \cdot (2x+1)$

$= \boxed{\dfrac{2x+1}{4(x^2+x)^{3/4}}}$

f) $y = (1+3x+4x^2)^{-3}$

$y' = -3(1+3x+4x^2)^{-4} \cdot (3+8x)$

$= \dfrac{-3(3+8x)}{(1+3x+4x^2)^4}$

$= \boxed{\dfrac{-9-24x}{(1+3x+4x^2)^4}}$

g) $y = \dfrac{1}{(x^3+2x^2+1)^2}$ $= (x^3+2x^2+1)^{-2}$

$y' = -2(x^3+2x^2+1)^{-3}\cdot(3x^2+4x)$

$\boxed{y' = \dfrac{-2(3x^2+4x)}{(x^3+2x^2+1)^3}}$

h) $y = \dfrac{4}{\sqrt{9-x^2}}$ $= 4\left[\sqrt{9-x^2}\right]^{-1}$

$= 4(9-x^2)^{-\frac{1}{2}}$

$y' = 4\cdot\left(-\frac{1}{2}(9-x^2)^{-3/2}\cdot -2x\right)$

$= -2(9-x^2)^{-3/2}\cdot -2x$

$= \boxed{\dfrac{4x}{(9-x^2)^{3/2}}}$

i) $y = (1+2\sqrt{x})^6$

$y' = 6(1+2x^{\frac{1}{2}})^5\cdot\left(\frac{1}{2}\cdot 2x^{-\frac{1}{2}}\right)$

$= 6(1+2x^{\frac{1}{2}})^5\cdot\dfrac{1}{x^{\frac{1}{2}}}$

$= \boxed{\dfrac{6(1+2\sqrt{x})^5}{\sqrt{x}}}$

j) $y = \sqrt{x+\sqrt{x}}$ $= (x+\sqrt{x})^{\frac{1}{2}}$

$y' = \frac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \dfrac{1}{2(x+\sqrt{x})^{\frac{1}{2}}}\cdot\left(1+\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2(x+\sqrt{x})^{\frac{1}{2}}}\cdot\dfrac{2\sqrt{x}+1}{2\sqrt{x}}$

$= \boxed{\dfrac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}}$

k) $y = x - \sqrt[5]{1+x^5-6x^{10}}$

$= x-(1+x^5-6x^{10})^{\frac{1}{5}}$

$y' = 1-\frac{1}{5}(1+x^5-6x^{10})^{-4/5}\cdot(5x^4-60x^9)$

$= 1-\dfrac{(5x^4-60x^9)}{5(1+x^5-6x^{10})^{4/5}}$ $= 1-\dfrac{\cancel{5}(x^4-12x^9)}{\cancel{5}(1+x^5-6x^{10})^{4/5}}$

$= \boxed{1-\dfrac{(x^4-12x^9)}{(1+x^5-6x^{10})^{4/5}}}$

l) $y = x^2+(x^2-1)^5$

$y' = 2x+5(x^2-1)^4\cdot 2x$

$= \boxed{2x+10x(x^2-1)^4}$

2. If $y = u^4+5u^2$, where $u = x^5+2x^2+1$, find $\dfrac{dy}{dx}$. Leave your answer in terms of u and x.

$\dfrac{dy}{du} = 4u^3+10u$ $\dfrac{du}{dx} = 5x^4+4x$

$\dfrac{dy}{dx} = (4u^3+10u)(5x^4+4x)$

$\dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{dy}{dx}$

3. Find $\left.\frac{dy}{dx}\right]_{x=4}$ if $y = u^2 - 2u^5$ and $u = x - \sqrt{x}$.

$u = x - \sqrt{x}$

$\frac{dy}{du} = 2u - 10u^4$

$\frac{du}{dx} = 1 - \frac{1}{2x^{\frac{1}{2}}}$

$\frac{dy}{dx} = (2u - 10u^4)\left(1 - \frac{1}{2\sqrt{x}}\right)$

$= \left[2(x-\sqrt{x}) - 10(x-\sqrt{x})^4\right]\left[1 - \frac{1}{2\sqrt{x}}\right]$ at $x = 4$

$= \left[2(4-2) - 10(4-2)^4\right]\left[1 - \frac{1}{4}\right] \rightarrow (4-160)(3/4) = \boxed{-117}$

4. Find $\left.\frac{dy}{dt}\right]_{t=1}$ if $y = \sqrt{1+r^2}$ and $r = \frac{t+1}{2t+1}$.

$\frac{dy}{dr} = \frac{1}{2}(1+r^2)^{-\frac{1}{2}} \cdot 2r$

$= \frac{r}{\sqrt{1+r^2}}$

$\frac{dr}{dt} = \frac{(2t+1)(1) - (t+1)(2)}{(2t+1)^2}$

$= \frac{2t+1-2t-2}{(2t+1)^2}$

$= \frac{-1}{(2t+1)^2}$

$\frac{dy}{dt} = \frac{dy}{dr} \cdot \frac{dr}{dt}$ at $t = 1$

$= \frac{r}{\sqrt{1+r^2}} \cdot \frac{-1}{(2t+1)^2}$

$= \frac{\frac{t+1}{2t+1}}{\sqrt{1+\left(\frac{t+1}{2t+1}\right)^2}} \cdot \frac{-1}{(2t+1)^2}$ $\quad \frac{\frac{2}{3}}{\sqrt{\frac{13}{9}}} \cdot \frac{-1}{9}$

$= \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{\sqrt{13}} \cdot \frac{-1}{9} = \boxed{\frac{-2}{9\sqrt{13}}}$

5. Find $\left.\frac{ds}{dt}\right]_{t=4}$ if $s = v + \frac{50}{v}$ and $v = 3t - \sqrt{t}$.

$\frac{ds}{dv} = 1 + \frac{-50}{v^2}$

$\frac{dv}{dt} = 3 - \frac{1}{2\sqrt{t}}$

$\frac{ds}{dv} \cdot \frac{dv}{dt}$ at $t = 4$

$\left(1 - \frac{50}{v^2}\right)\left(3 - \frac{1}{2\sqrt{t}}\right)$

$= \left(1 - \frac{50}{(3t-\sqrt{t})^2}\right)\left(3 - \frac{1}{2\sqrt{t}}\right) \rightarrow \left(1 - \frac{50}{100}\right)\left(3 - \frac{1}{4}\right) = \left(\frac{1}{2}\right)\left(\frac{11}{4}\right) = \boxed{\frac{11}{8}}$

6. Differentiate

a) $F(x) = x\sqrt{x^2+1}$

Need Product and Chain Rule

$F'(x) = x(x^2+1)^{\frac{1}{2}}$

$x\left(\frac{1}{2(x^2+1)^{\frac{1}{2}}} \cdot 2x\right) + \sqrt{x^2+1}\,(1)$

$\frac{2x^2}{2\sqrt{(x^2+1)}} + \sqrt{(x^2+1)}\left(\frac{\sqrt{(x^2+1)}}{\sqrt{(x^2+1)}}\right)$

$\frac{x^2 + x^2 + 1}{\sqrt{x^2+1}} = \boxed{\frac{2x^2+1}{\sqrt{x^2+1}}}$

b) $F(x) = (2x+1)(4x-1)^5$

$F'(x) = 5(4x-1)^4(4)(2x+1) + (4x-1)^5(2)$

$= 20(4x-1)^4(2x+1) + 2(4x-1)^5$ Factor out $(4x-1)^4$.

$= [20(2x+1) + 2(4x-1)](4x-1)^4$

$= [40x + 20 + 8x - 2](4x-1)^4$

$(48x+18)(4x-1)^4$

$\boxed{6(8x+3)(4x-1)^4}$

c) $G(x) = (x^2-1)^4(2-3x)$

$G'(x) = (x^2-1)^4(-3) + (2-3x)(4)(x^2-1)^3(2x)$

$= -3(x^2-1)^4 + 8x(2-3x)(x^2-1)^3$ Factor out $(x^2-1)^3$

$= (x^2-1)^3[-3(x^2-1) + 8x(2-3x)]$

$= (x^2-1)^3[-3x^2+3+16x-24x^2]$

$\boxed{(x^2-1)^3(-27x^2+16x+3)}$

d) $G(x) = (x^4-x+1)^2(x^2-2)^3$

$G'(x) = (x^4-x+1)^2(3)(x^2-2)^2 \cdot 2x + (x^2-2)^3(2(x^4-x+1)(4x^3-1))$

$= 6x(x^4-x+1)^2(x^2-2)^2 + (x^2-2)^3(2(x^4-x+1)(4x^3-1))$

$(x^4-x+1)(x^2-2)^2[6x(x^4-x+1) + 2(x^2-2)(4x^3-1)]$

$(x^4-x+1)(x^2-2)^2[6x^5-6x^2+6x + 8x^5-2x^2-16x^3+4]$

$(x^4-x+1)(x^2-2)^2$
$(14x^5-16x^3-8x^2+6x+4)$

or

$\boxed{2(x^4-x+1)(x^2-2)^2(7x^5-8x^3-4x^2+3x+2)}$

e) $F(x) = \dfrac{x}{\sqrt{2x+3}}$ Quotient Rule and Chain

$F'(x) = \dfrac{\sqrt{2x+3}(1) - [x(\frac{1}{2\sqrt{2x+3}}) \cdot 2]}{\sqrt{2x+3}^2}$

$= \sqrt{2x+3} - \dfrac{2x}{2\sqrt{2x+3}}$

$\dfrac{\sqrt{2x+3}^2}{\sqrt{2x+3}} - \dfrac{x}{\sqrt{2x+3}}$

$\dfrac{2x+3-x}{\sqrt{2x+3}^2} = \boxed{\dfrac{x+3}{\sqrt{2x+3}^2}}$

f) $f(t) = \dfrac{(1+2t)^5}{(3t^2-5)^2}$

$$f'(t) = \frac{(3t^2-5)^2 \cdot 5(1+2t)^4 \cdot 2 - [2(3t^2-5) \cdot 6t \cdot (1+2t)^5]}{(3t^2-5)^4}$$

$$= \frac{10(3t^2-5)^2(1+2t)^4 - 12t(3t^2-5)(1+2t)^5}{(3t^2-5)^4}$$

$$\frac{2\cancel{(3t^2-5)}(1+2t)^4[5(3t^2-5) - 6t(1+2t)]}{(3t^2-5)^{\cancel{4}\,3}}$$

$$\frac{2(1+2t)^4(15t^2-25-6t-12t^2)}{(3t^2-5)^3}$$

$$\boxed{\frac{2(1+2t)^4(3t^2-6t-25)}{(3t^2-5)^3}}$$

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g) $g(x) = \left(\dfrac{x+2}{x-2}\right)^3$

$$g'(x) = 3\left(\frac{x+2}{x-2}\right)^2 \cdot \frac{(x-2)(1) - (x+2)(1)}{(x-2)^2}$$

$$= 3\left(\frac{x+2}{x-2}\right)^2 \cdot \frac{[x-2-x-2]}{(x-2)^2}$$

$$3\left(\frac{x+2}{x-2}\right)^2 \cdot \frac{-4}{(x-2)^2}$$

$$= \boxed{\frac{-12(x+2)^2}{(x-2)^4}}$$

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h) $h(t) = \left(\dfrac{t^2+1}{t+1}\right)^{10}$

$$h'(t) = 10\left(\frac{t^2+1}{t+1}\right)^9 \cdot \frac{(t+1)(2t) - (t^2+1)(1)}{(t+1)^2}$$

$$= 10\frac{(t^2+1)^9}{(t+1)^9} \cdot \frac{2t^2+2t-t^2-1}{(t+1)^2}$$

$$\boxed{\frac{10(t^2+1)^9(t^2+2t-1)}{(t+1)^{11}}}$$

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i) $y = \sqrt{\dfrac{x^2-1}{x^2+1}}$ $= \left(\dfrac{(x^2-1)}{(x^2+1)}\right)^{\frac{1}{2}}$

$$y' = \frac{1}{2}\left(\frac{(x^2-1)}{(x^2+1)}\right)^{-\frac{1}{2}} \cdot \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{1}{2}\left(\frac{(x^2+1)}{(x^2-1)}\right)^{\frac{1}{2}} \cdot \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$$\frac{\sqrt{(x^2+1)}}{2\sqrt{(x^2-1)}} \cdot \frac{4x}{(x^2+1)^2}$$

$$\frac{(x^2+1)^{\frac{1}{2}} \cdot 4x}{2\sqrt{x^2-1} \cdot (x^2+1)^{4/2}}$$

$$\boxed{= \frac{2x}{\sqrt{x^2-1}\,(x^2+1)^{3/2}}}$$

j) $y = \frac{(2x+3)^3}{\sqrt{4x-7}}$

$$y' = \frac{\sqrt{4x-7}\,(3)(2x+3)^2(2) - (2x+3)^3 \frac{1}{2\sqrt{4x-7}} \cdot 4}{4x-7}$$

$$= \frac{6(2x+3)^2(4x-7) - 2(2x+3)^3}{(4x-7)\sqrt{4x-7}} \to \frac{2(2x+3)^2[3(4x-7) - (2x+3)]}{(4x-7)^{3/2}}$$

$$\to \frac{2(2x+3)^2(10x-24)}{(4x-7)^{3/2}} \to \boxed{\frac{4(2x+3)^2(5x-12)}{(4x-7)^{3/2}}}$$

k) $y = 3\sqrt{x}(2x+1)^5 + \sqrt{4x-3}$

$$y' = 3\left[\sqrt{x}\cdot 5(2x+1)^4\cdot 2 + (2x+1)^5 \frac{1}{2\sqrt{x}}\right] + \frac{1}{2\sqrt{4x-3}}\cdot 4$$

$$= 3\left[10\sqrt{x}(2x+1)^4 + \frac{(2x+1)^5}{2\sqrt{x}}\right] + \frac{2}{\sqrt{4x-3}}$$

$$= 3\left[\frac{20x(2x+1)^4 + (2x+1)^5}{2\sqrt{x}}\right] + \frac{2}{\sqrt{4x-3}} \to \frac{3(2x+1)^4[20x+2x+1]}{2\sqrt{x}} + \frac{2}{\sqrt{4x-3}}$$

$$= \boxed{\frac{3(2x+1)^4(22x+1)}{2\sqrt{x}} + \frac{2}{\sqrt{4x-3}}}$$

l) $y = \sqrt{1+\sqrt[3]{x}} = (1+\sqrt[3]{x})^{\frac{1}{2}}$

$$y' = \frac{1}{2}(1+x^{\frac{1}{3}})^{-\frac{1}{2}} \cdot \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{2(1+x^{\frac{1}{3}})^{\frac{1}{2}}} \cdot \frac{1}{3x^{2/3}} \to \boxed{\frac{1}{6x^{2/3}\sqrt{1+\sqrt[3]{x}}}}$$

m) $y = \left(t + \sqrt[3]{t+t^2}\right)^{20} = \left(t + (t+t^2)^{\frac{1}{3}}\right)^{20}$

$$y' = 20\left(t + (t+t^2)^{\frac{1}{3}}\right)^{19} \cdot \left(1 + \frac{1}{3}(t+t^2)^{-2/3}\cdot(1+2t)\right)$$

Double chain Rule

$$= 20\left(t+\sqrt[3]{t+t^2}\right)^{19} \cdot \left(1 + \frac{1+2t}{3(t+t^2)^{2/3}}\right)$$

$$= \boxed{20\left(t+\sqrt[3]{t+t^2}\right)^{19}\left(\frac{3(t+t^2)^{2/3} + 1 + 2t}{3(t+t^2)^{2/3}}\right)}$$

n) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$ $= \left(x+\sqrt{x+\sqrt{x}}\right)^{\frac{1}{2}} \to \left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$y' = \frac{1}{2\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{\frac{1}{2}}} \cdot \left(1+\frac{1}{2(x+x^{\frac{1}{2}})^{\frac{1}{2}}} \cdot \left(1+\frac{1}{2x^{\frac{1}{2}}}\right)\right)$$

$$= \boxed{\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left(1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)}$$

7. Find the equation of the tangent line to the curve $y=(x^2-3)^8$ at the point $(2,1)$.

$y' = 8(x^2-3)^7\cdot(2x)$

$y' = 16x(x^2-3)^7$ at $x=2$

$y' = 16(2)(4-3)^7$

$= \boxed{32}$

$y = 32x+b$

$1 = 32(2)+b$

$-63 = b$

$\boxed{y = 32x-63}$

8. Find the equation of the tangent line to the curve $y=\dfrac{1}{\sqrt{20-x^4}}$ at the point $\left(2,\frac{1}{2}\right)$

$y = (20-x^4)^{-\frac{1}{2}}$

$y' = -\dfrac{1}{2(20-x^4)^{3/2}}\cdot -4x^3$

$\dfrac{4x^3}{2(20-x^4)^{3/2}}$

$\dfrac{2x^3}{(20-x^4)^{3/2}}$ at $x=2$

$\dfrac{2(2)^3}{(20-16)^{3/2}} = \dfrac{16}{8} = \boxed{2}$

$y = 2x+b$

$\frac{1}{2} = 2(2)+b$

$\frac{1}{2} = 4+b$

$-\frac{7}{2} = b$

$\boxed{y = 2x-\frac{7}{2}}$

9. If $F(x)=f(g(x))$, where $g(2)=4$, $g'(2)=3$, and $f'(4)=5$, find $F'(2)$.

$F'(x) = f'(g(x))\cdot g'(x)$

$= f'\,g(2)\cdot g'(2)$

$= f'(4)\cdot 3$

$= 5\cdot 3$

$= \boxed{15}$

10. If $G(x) = h(p(x))$, where $h(5) = 1, h'(5) = 2, h'(1) = 3, p(1) = 5$, and $p'(1) = 7$, find $G'(1)$.

$G'(x) = h'(p(x)) \cdot p'(x)$

$h'(p(1)) \cdot p'(1)$

$h'(5) \cdot 7$

$2 \cdot 7$

$\boxed{14}$

11. If f is a differentiable function, find expressions for the derivatives of the following functions.

a) $F(x) = f(x^4)$

$f'(x^4) \cdot 4x^3$

$= 4x^3 f'(x^4)$

b) $G(x) = [f(x)]^4$

$4[f(x)]^3 \cdot f'(x)$

c) $H(x) = f(\sqrt{x})$

$f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

$\frac{1 \cdot f'(\sqrt{x})}{2\sqrt{x}}$

d) $P(x) = \sqrt{f(x)}$

$(f(x))^{\frac{1}{2}}$

$\frac{1}{2\sqrt{f(x)}} \cdot f'(x)$

$\frac{f'(x)}{2\sqrt{f(x)}}$

e) $y = f(f(x))$

$f'(f(x)) \cdot f'(x)$

f) $y = \sqrt{1 + [f(x)]^2}$

$(1+(f(x))^2)^{\frac{1}{2}}$

$\frac{1}{2(1+(f(x))^2)^{\frac{1}{2}}} \cdot 2f(x) \cdot f'(x)$

$\frac{\cancel{2} f(x) f'(x)}{\cancel{2}(\sqrt{1+(f(x))^2}} = \frac{f(x) f'(x)}{\sqrt{1+(f(x))^2}}$

g) $y = [f(x^2)]^2$

$y' = 2[f(x^2)] \cdot f'(x^2) \cdot 2x$

$4x[f(x^2)]f'(x^2)$

h) $y = f([f(x)]^3)$

$y' = f'([f(x)]^3) \cdot 3[f(x)]^2 \cdot f'(x)$

$3f'(x)[f(x)]^2 f'([f(x)]^3)$

12.

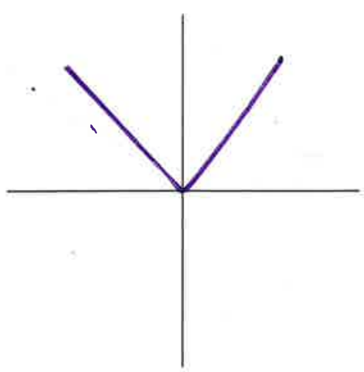
a) Use the Chain Rule and the fact that $|x| = \sqrt{x^2}$ to show that $\frac{d}{dx}|x| = \frac{x}{|x|}$

if $|x| = \sqrt{x^2}$

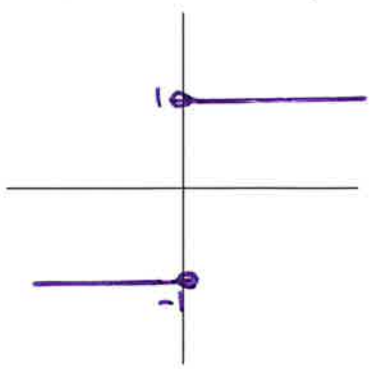
$$\frac{d}{dx}|x| = \frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

b) Sketch the graphs of the function $f(x) = |x|$ and its derivative.

f



f'



c) Use the result in part (a) to differentiate the function $g(x) = x|x|$

Product Rule $\frac{d}{dx}|x| = \frac{x}{|x|}$

$$= x \cdot \frac{x}{|x|} + |x|$$

$$= \frac{x^2}{|x|} + \frac{|x|^2}{|x|} \rightarrow \frac{x^2 + |x|^2}{|x|} = \frac{2|x|^2}{|x|} = 2|x|$$