2.5 The Quotient Rule

The derivative of a quotient of two functions can be found using the following rule.

Quotient Rule

If both f and g are differentiable, then so is the quotient

$$F(x) = \frac{f(x)}{g(x)}$$

And

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Proof

Since

$$F(x) = \frac{f(x)}{g(x)}$$

We have

$$f(x) = F(x)g(x)$$

So, we can now use the Product Rule,

$$f'(x) = F(x)g'(x) + F'(x)g(x)$$

Now all we need to do is solve the above equation for $F'(x)$:

$$\begin{aligned} F'(x)g(x) &= f'(x) - F(x)g'(x) \\ &= f'(x) - \frac{f(x)}{g(x)}g'(x) \\ F'(x) &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}} \end{aligned}$$

Leibniz Notation for the Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

It is important to note that the *order* of the terms is the key to calculating the correct derivative. Unlike the product rule, because of the minus sign in the numerator. In words, the Quotient Rule says the *bottom function multiplies by the derivative of the top minus the top function multiplied by the derivative of the bottom function all divided by the bottom function squared.*

Ex. 1

Differentiate the following function.

$F(x) = \dfrac{x^2 + 2x - 3}{x^3 + 1}$ ← $f(x)$ ← $g(x)$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{(x^3+1)(2x+2) - [(x^2+2x-3)(3x^2)]}{(x^3+1)^2}$$

$$= \frac{2x^4+2x^3+2x+2 - [3x^4+6x^3-9x^2]}{(x^3+1)^2}$$

$$\frac{2x^4+2x^3+2x+2-3x^4-6x^3+9x^2}{(x^3+1)^2}$$

$$F'(x) = \frac{-x^4-4x^3+9x^2+2x+2}{(x^3+1)^2}$$

Ex. 2

Find $\frac{dy}{dx}$ if

$y = \dfrac{\sqrt{x}}{1+2x}$ ← $f(x)$ ← $g(x)$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{(1+2x)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(2)}{(1+2x)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}} + \frac{2x}{2\sqrt{x}} - 2\sqrt{x}}{(1+2x)^2}$$

$$\frac{\frac{1}{2\sqrt{x}} + \sqrt{x} - 2\sqrt{x}}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}} - \sqrt{x}}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1-2x}{2\sqrt{x}}}{(1+2x)^2} \rightarrow \frac{1-2x}{2\sqrt{x}(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{1-2x}{2\sqrt{x}(1+2x)^2}$$

Homework Assignment

- Exercise 2.5: #1 – 3odd, 4 – 8