$$\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Exercise 2.5 – Practice Problems

1. Differentiate

a) $f(x) = \frac{x-1}{x+1}$

$$\frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

$$\frac{x+1-x+1}{(x+1)^2} \rightarrow \boxed{\frac{2}{(x+1)^2}}$$

b) $f(x) = \frac{2x-1}{x^2+1}$

$$\frac{(x^2+1)(2) - [(2x-1)(2x)]}{(x^2+1)^2}$$

$$\frac{2x^2+2-4x^2+2x}{(x^2+1)^2} \rightarrow \boxed{\frac{-2x^2+2x+2}{(x^2+1)^2}}$$

c) $g(x) = \frac{x}{x^2+2x-1}$

$$\frac{(x^2+2x+1)(1) - (x)(2x+2)}{(x^2+2x-1)^2}$$

$$\frac{x^2+2x+1-2x^2-2x}{(x^2+2x-1)^2} \rightarrow \boxed{\frac{-x^2+1}{(x^2+2x-1)^2}}$$

d) $g(x) = \frac{x^3-1}{x^2+x+1}$

$$\frac{(x^2+x+1)(3x^2) - [(x^3-1)(2x+1)]}{(x^2+x+1)^2}$$

$$\frac{3x^4+3x^3+3x^2-2x^4-x^3+2x+1}{(x^2+x+1)^2}$$

$$\frac{x^4+2x^3+3x^2+2x+1}{(x^2+x+1)^2} = \boxed{1}$$

e) $y = \frac{\sqrt{x}}{x^2+1}$

$$\frac{(x^2+1)\left(\frac{1}{2\sqrt{x}}\right) - [\sqrt{x}(2x)]}{(x^2+1)^2}$$

$$\frac{\frac{x^2+1}{2\sqrt{x}} - 2x\sqrt{x}}{(x^2+1)^2} \rightarrow \frac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2}$$

$$\boxed{\frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}}$$

f) $y = \frac{\sqrt{x}+2}{\sqrt{x}-2}$

$$\rightarrow \frac{(\sqrt{x}-2)\left(\frac{1}{2\sqrt{x}}\right) - \left[(\sqrt{x}+2)\left(\frac{1}{2\sqrt{x}}\right)\right]}{(\sqrt{x}-2)^2}$$

$$\frac{\frac{\sqrt{x}-2}{2\sqrt{x}} - \frac{(\sqrt{x}+2)}{2\sqrt{x}}}{(\sqrt{x}-2)^2} \rightarrow \frac{\frac{\sqrt{x}-2-\sqrt{x}-2}{2\sqrt{x}}}{(\sqrt{x}-2)^2}$$

$$\boxed{\frac{-2}{2\sqrt{x}(\sqrt{x}-2)^2}}$$

g) $f(t) = \frac{2t+1}{t^2-3t+4}$

$$\frac{(t^2-3t+4)(2) - [(2t+1)(2t-3)]}{(t^2-3t+4)^2}$$

$$\frac{2t^2-6t+8-[4t^2-6t+2t-3]}{(t^2-3t+4)^2}$$

$$\frac{2t^2-6t+8-4t^2+4t+3}{(t^2-3t+4)^2}$$

$$\boxed{\frac{-2t^2-2t+11}{(t^2-3t+4)^2}}$$

h) $g(t) = \frac{2t^2+3t+1}{t-1}$

$$\frac{(t-1)(4t+3) - [2t^2+3t+1]}{(t-1)^2}$$

$$\frac{4t^2-t-3-2t^2-3t-1}{(t-1)^2}$$

$$\boxed{\frac{2t^2-4t-4}{(t-1)^2}}$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

i) $f(x) = \dfrac{1}{x^4 - x^2 + 1}$

$$\frac{(x^4-x^2+1)(0) - [4x^3-2x]}{(x^4-x^2+1)^2}$$

$$\frac{0-4x^3+2x}{(x^4-x^2+1)^2} \rightarrow \boxed{\frac{-4x^3+2x}{(x^4-x^2+1)^2}}$$

j) $f(x) = \dfrac{ax+b}{cx+d}$

$$\frac{(cx+d)(a) - [(ax+b)(c)]}{(cx+d)^2}$$

$$\frac{acx^2+da-acx^2-bc}{(cx+d)^2} \rightarrow \boxed{\frac{ad-bc}{(cx+d)^2}}$$

k) $f(x) = \dfrac{x^6}{x^5 - 10}$

$$\frac{(x^5-10)6x^5 - [x^6(5x^4)]}{(x^5-10)^2}$$

$$\frac{6x^{10}-60x^5-5x^{10}}{(x^5-10)^2} \rightarrow \boxed{\frac{x^{10}-60x^5}{(x^5-10)^2}}$$

l) $f(x) = \dfrac{1-\frac{1}{x}}{x+1} \rightarrow \dfrac{1-x^{-1}}{x+1}$

$$\frac{(x+1)(x^{-2}) - (1-x^{-1})(1)}{(x+1)^2} \rightarrow \frac{x^{-1}+x^{-2}-1+x^{-1}}{(x+1)^2}$$

$$\frac{x^{-2}+2x^{-1}-1}{(x+1)^2} \leftarrow \text{Factor out } x^{-2}$$

$$\rightarrow \frac{x^{-2}(1+2x-x^2)}{(x+1)^2} = \boxed{\frac{1+2x-x^2}{x^2(x+1)^2}}$$

2. Find the Domain of f and compute its derivative.

a) $f(x) = \dfrac{2+x}{1-2x}$ $D_f: \{x \mid x \in \mathbb{R}, x \neq \frac{1}{2}\}$

$$f'(x) = \frac{(1-2x)(1) - [(2+x)(-2)]}{(1-2x)^2}$$

$$= \frac{1-2x+4+2x}{(1-2x)^2} \rightarrow \boxed{\frac{5}{(1-2x)^2}}$$

b) $f(x) = \dfrac{x}{x^2-1}$ $D_f = \{x \mid x \in \mathbb{R}, x \neq \pm 1\}$

$$\frac{(x^2-1)(1) - (2x)(x)}{(x^2-1)^2}$$

$$\frac{x^2-1-2x^2}{(x^2-1)^2} \rightarrow \boxed{\frac{-x^2-1}{(x^2-1)^2}}$$

c) $f(x) = \dfrac{1}{(x+1)(2x-3)} \rightarrow \dfrac{1}{2x^2-x-3}$

$$\frac{(2x^2-x-3)(0) - [1(4x-1)]}{(2x^2-x-3)^2}$$

$$\frac{0-4x+1}{(2x^2-x-3)^2} \rightarrow \frac{-4x+1}{(2x^2-x-3)^2}$$

$D_f \{x \mid x \in \mathbb{R}, x \neq -1, x \neq \frac{3}{2}\}$

d) $f(x) = \dfrac{2x+1}{x^2+2x-3}$ $D_f \{x \mid x \in \mathbb{R}, x \neq -3, x \neq +1\}$

$$\frac{(x^2+2x-3)(2) - [(2x+1)(2x+2)]}{(x^2+2x-3)^2}$$

$$\frac{2x^2+4x-6-[4x^2+6x+2]}{(x^2+2x-3)^2}$$

$$\frac{2x^2+4x-6-4x^2-6x-2}{(x^2+2x-3)^2}$$

$$\boxed{\frac{-2x^2-2x-8}{(x^2+2x-3)^2}}$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$D_f \{x \mid x \in \mathbb{R}, x \neq \pm 1\}$

e) $f(x) = \dfrac{x^2 + 2x}{x^4 - 1}$

$$\frac{(x^4-1)(2x+2) - [(x^2+2x)(4x^3)]}{(x^4-1)^2}$$

$$\rightarrow \frac{2x^5+2x^4-2x-2-4x^5-8x^4}{(x^4-1)^2}$$

$$\boxed{\frac{-2x^5-6x^4-2x-2}{(x^4-1)^2}}$$

$D_f \{x \mid x \in \mathbb{R}, x \geq 0, x \neq 9\}$

f) $f(x) = \dfrac{x^2}{\sqrt{x} - 3}$

$$\frac{(\sqrt{x}-3)(2x) - [x^2(\frac{1}{2\sqrt{x}})]}{(\sqrt{x}-3)^2}$$

$$\frac{2x^{3/2} - 6x - \frac{x^2}{2\sqrt{x}}}{(\sqrt{x}-3)^2} \rightarrow \frac{4x^{3/2} - 12x - x^{3/2}}{2(\sqrt{x}-3)^2}$$

$$\boxed{\frac{3x^{3/2} - 12x}{2(\sqrt{x}-3)^2}}$$

3. Find an equation of the tangent line to the curve at the given point.

a) $y = \dfrac{x}{x-2}, (4, 2)$

$$\frac{(x-2)(1) - [x(1)]}{(x-2)^2} \rightarrow \frac{x-2-x}{(x-2)^2}$$

$y' = \dfrac{-2}{(x-2)^2}$ at $x = 4$ $\quad \dfrac{-2}{2^2} = \boxed{-\frac{1}{2}}$

$y = -\frac{1}{2}x + b$

$2 = -\frac{1}{2}(4) + b$

$4 = b$

$$\boxed{y = -\frac{1}{2}x + 4}$$

b) $y = \dfrac{1+3x}{2-3x}, (1, -4)$

$$\frac{(2-3x)(3) - [(1+3x)(-3)]}{(2-3x)^2}$$

$$\frac{6-9x+3+9x}{(2-3x)^2} = \frac{9}{(2-3x)^2}$$

at $x = 1$

$\frac{9}{1} = 9$

$y = 9x + b$

$-4 = 9(1) + b$

$-13 = b$

$$\boxed{y = 9x - 13}$$

c) $y = \dfrac{1}{x^2+1}, \left(-2, \frac{1}{5}\right)$

$$\frac{(x^2+1)(0) - [1(2x)]}{(x^2+1)^2}$$

$$\frac{0-2x}{(x^2+1)^2} \rightarrow \frac{-2x}{(x^2+1)^2}$$ at $x = -2$

$\frac{4}{25}$

$y = \frac{4}{25}x + b$

$\frac{1}{5} = \frac{4(-2)}{25} + b$

$\frac{5}{25} = \frac{-8}{25} + b$

$b = \frac{13}{25}$

$$\boxed{y = \frac{4}{25}x + \frac{13}{25}}$$

d) $y = \dfrac{x^3 - 1}{1 + 2x^2}, (1, 0)$

$$\frac{(1+2x^2)(3x^2) - [(x^3-1)(4x)]}{(1+2x^2)^2}$$

$$\frac{3x^2+6x^4-4x^4+4x}{(1+2x^2)^2} = \frac{2x^4+3x^2+4x}{(1+2x^2)^2}$$

at $x = 1$

$\frac{9}{9} = 1$

$y = x + b$

$0 = 1 + b$

$b = -1$

$$\boxed{y = x - 1}$$

4. If $f(2) = 3, f'(2) = 5, g(2) = -1, and\ g'(2) = -4$ find $\left(\frac{f}{g}\right)'(2)$.

$$\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2}$$

$$= \frac{-1(5) - (3)(-4)}{(-1)^2} = \frac{-5+12}{1} = \boxed{7}$$

5. Show that there are no tangents to the curve $y = \frac{x+2}{3x+4}$ with a positive slope.

$$y' = \frac{(3x+4)(1) - [(x+2)(3)]}{(3x+4)^2}$$

$$= \frac{3x+4-3x-6}{(3x+4)^2}$$

$$= \frac{-2}{(3x+4)^2}$$ ← since denominator is squared no x-values result in a negative denominator

$\boxed{\text{So, } \frac{-}{+} = - \text{ always}}$

6. At what points on the curve $y = \frac{x^2}{2x+5}$ is the tangent line horizontal?

↑ $m = 0$

$$\frac{(2x+5)(2x) - [x^2(2)]}{(2x+5)^2}$$

$$\frac{4x^2+10x-2x^2}{(2x+5)^2}$$

$$\frac{2x^2+10x}{(2x+5)^2}$$ → only numerator maters

$2x^2+10x = 0$

$2x(x+5) = 0$

$x = 0 \quad x = -5$

if $x = 0$, $y = 0$: $\boxed{(0,0)}$

$x = -5$, $y = -5$: $\boxed{(-5,-5)}$

7. Find the points on the curve $y = \frac{x}{x-1}$ where the tangent line is parallel to the line $x + 4y = 1$

$y' = \frac{(x-1)(1) - [x(1)]}{(x-1)^2}$

$4y = -x + 1$

$y = -\frac{1}{4}x + \frac{1}{4}$

$m = -\frac{1}{4}$

$\frac{x-1-x}{(x-1)^2}$

$\frac{-1}{(x-1)^2} = -\frac{1}{4}$

Since numerators are equal make denominators equal

$\frac{-1}{(x-1)^2} = m$

$(x-1)^2 = 4$

$x - 1 = \pm 2 \rightarrow x - 1 = 2 \quad x = 3$

$x - 1 = -2 \quad x = -1$

if $x = 3$, $y = 3/2$ $\boxed{(3, 3/2)}$

if $x = -1$, $y = \frac{1}{2}$ $\boxed{(-1, \frac{1}{2})}$

8. If f is a differentiable function, find expressions for the derivatives of the following functions.

a) $y = \frac{1}{f(x)}$

$y' = \frac{f(x)(0) - 1(f'(x))}{(f(x))^2}$

$= \frac{0 - f'(x)}{(f(x))^2}$

$= \boxed{\frac{-f'(x)}{(f(x))^2}}$

b) $y = \frac{f(x)}{x}$

$y' = \frac{xf'(x) - [f(x)(1)]}{x^2}$

$= \boxed{\frac{xf'(x) - f(x)}{x^2}}$

c) $y = \frac{x}{f(x)}$

$y' = \frac{f(x)(1) - [xf'(x)]}{[f(x)]^2}$

$= \boxed{\frac{f(x) - xf'(x)}{[f(x)]^2}}$

9. In Section 2.2 we proved the Power Rule for positive integer exponents. Use the Quotient Rule to deduce the Power Rule for the case of negative integer exponents; that is, prove that

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

Where n is a positive integer.

Proof:

$\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\frac{1}{x^n}$

$\frac{x^n(0) - [1(nx^{n-1})]}{(x^n)^2}$

$\frac{0 - nx^{n-1}}{x^{2n}} = (-nx^{n-1})(x^{-2n})$

$= -nx^{n-1-2n}$

$= -nx^{-n-1}$ ✓