2.4 The Product Rule

Is the derivative of a product the product of its derivatives? This is what Leibniz guessed three centuries ago. Test it with the example below.

Ex.

Let $f(x) = x$ and $g(x) = x^2$. Test whether $(fg)' = f'g'$.

$(fg)'$ is $\quad f'g'$ is $\quad f'(x) = 1 \quad g'(x) = 2x$

$f \cdot g = x \cdot x^2 = x^3 \quad f'g' = 2x$

$(fg)' = 3x^2$ NOT THE SAME.

Leibniz eventually arrived at the correct formula, which we call the Product Rule.

* **Product Rule** *

If both f and g are differentiable, then so is fg and

$$(fg)' = fg' + f'g$$

In Leibniz notation:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Proof

Let $F = fg$, that is, $F(x) = (fg)(x) = f(x)g(x)$.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Like the proof of the Sum Rule, the next step is to separate the functions f and g. This can be accomplished by subtracting and adding the term $f(x+h)g(x)$ in the numerator. We can then factor as follows.

$$F'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right]$$
$$= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$\boxed{= f(x)g'(x) + g(x)f'(x)}$$

Notes:

- Since $g(x)$ is a constant with respect to h
$$\lim_{h\to 0} g(x) = g(x)$$
- Since $f(x)$ is continuous and differentiable functions are continuous
$$\lim_{h\to 0} f(x+h) = f(x)$$

call this $f(x)$ call this $g(x)$

You could expand and simplify but why do more work than you have to.

Ex. 1
Find $\frac{dy}{dx}$ if $y = (2x^3 + 5)(3x^2 - x)$

$(fg)' = fg' + f'g$

$(2x^3+5)(6x-1) + 6x^2(3x^2-x)$

$12x^4 - 12x^3 + 30x - 5 + 18x^4 - 6x^3$

$\boxed{\frac{dy}{dx} = 30x^4 - 18x^3 + 30x - 5}$

Ex. 2
Differentiate $f(x) = \sqrt{x}(2 - 3x)$

$f(x)$ $g(x)$

$(fg)' = fg' + f'g$

$\sqrt{x}(-3) + \frac{1}{2}x^{-\frac{1}{2}}(2-3x)$

$-3\sqrt{x} + \frac{2-3x}{2\sqrt{x}}$

$-3\sqrt{x} + \frac{2}{2\sqrt{x}} - \frac{3x}{2\sqrt{x}}$

$-3\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{3\sqrt{x}}{2}$

$-\frac{6\sqrt{x}}{2} - \frac{3\sqrt{x}}{2} + \frac{1}{\sqrt{x}}$

$\boxed{\frac{1}{\sqrt{x}} - \frac{9\sqrt{x}}{2}}$

Ex. 3
Find the slope of the tangent to the graph of the function $f(x) = (3x^2 + 2)(2x^3 - 1)$ at the point $(1, 5)$.

notation doesn't matter → $f(x)$ $g(x)$ could call it $h(x)$ and $g(x)$

$(fg)' = fg' + f'g$

$(3x^2+2)(6x^2) + 6x(2x^3-1)$

$18x^4 + 12x^2 + 12x^4 - 6x$

$\frac{dy}{dx} = 30x^4 + 12x^2 - 6x$

our new $f'(x)$

$f'(1) = 30(1)^4 + 12(1) - 6(1)$

$= 36$

slope is 36

$y = 36x + b$ at $(1,5)$

$5 = 36(1) + b$

$-31 = b$

$\boxed{y = 36x - 31}$

Homework Assignment

- Exercise 2.4: 1 – 3odd, 4 – 8