$(fg)' = fg' + f'g$

Exercise 2.4 – Practice Problems

1. Use the Product Rule to find the derivative. Do not simplify your answer.

a) $f(x) = \overset{f}{(2x-1)}\overset{g}{(x^2+1)}$

$\boxed{(2x-1)(2x) + 2(x^2+1)}$

b) $f(x) = \overset{f}{x}\overset{g}{(3x-8)}$

$x(3) + 1(3x-8)$

$\boxed{3x + (3x-8)}$

c) $y = \overset{f}{x^2}\overset{g}{(1+x-3x^2)}$

$x^2(-6x+1) + 2x(1+x-3x^2)$

$\boxed{x^2(1-6x) + 2x(1+x-3x^2)}$

d) $y = \overset{f}{(x^3+x^2+1)}\overset{g}{(x^2+2)}$

$\boxed{(x^3+x^2+1)(2x) + (3x^2+2x)(x^2+2)}$

e) $f(t) = \overset{f}{(t^4+t^2-1)}\overset{g}{(t^2-2)}$

$\boxed{(t^4+t^2-1)(2t) + (4t^3+2t)(t^2-2)}$

f) $f(t) = \overset{f}{\sqrt[3]{t}}\overset{g}{(1-t)}$

$\sqrt[3]{t}(-1) + \frac{1}{3}t^{-2/3}(1-t)$

$\boxed{-\sqrt[3]{t} + \frac{(1-t)}{3t^{2/3}}}$

g) $F(y) = \overset{f}{\sqrt{y}}\overset{g}{(y-2\sqrt{y}+2)}$

$\sqrt{y}(1-y^{-\frac{1}{2}}) + \frac{1}{2}y^{-\frac{1}{2}}(y-2\sqrt{y}+2)$

$\boxed{\sqrt{y}\left(1-\frac{1}{\sqrt{y}}\right) + \frac{(y-2\sqrt{y}+2)}{2\sqrt{y}}}$

h) $G(y) = \overset{f}{(y-y^2)}\overset{g}{\left(2y-y^{\frac{4}{3}}\right)}$

$\boxed{(y-y^2)\left(2-\frac{4}{3}y^{\frac{1}{3}}\right) + (1-2y)\left(2y-y^{\frac{4}{3}}\right)}$

$fg' + f'g$

2. Use the Product Rule to differentiate each function. Simplify your answer.

a) $y = \overset{f}{x^3}\overset{g}{(x^2 + 2x + 3)}$

$y' = x^3(2x+2) + 3x^2(x^2+2x+3)$

$= 2x^4 + 2x^3 + 3x^4 + 6x^3 + 9x^2$

$= \boxed{5x^4 + 8x^3 + 9x^2}$

b) $y = \overset{f}{x^{-2}}\overset{g}{(x^3 - 3x^2 + 6)}$

$y' = x^{-2}(3x^2 - 6x) + -2x^{-3}(x^3 - 3x^2 + 6)$

$= \frac{3x^2 - 6x}{x^2} - \frac{2(x^3 - 3x^2 + 6)}{x^3}$

$= \frac{3x^3 - 6x^2 - 2x^3 + 6x^2 - 12}{x^3} = \boxed{\frac{x^3 - 12}{x^3}}$

c) $f(x) = \overset{f}{(1 - x^2)}\overset{g}{(2 - x^3)}$

$f'(x) = (1-x^2)(-3x^2) + (-2x)(2-x^3)$

$\rightarrow -3x^2 + 3x^4 - 4x + 2x^4$

$= \boxed{5x^4 - 3x^2 - 4x}$

d) $f(x) = \overset{f}{(3x^3 + 4)}\overset{g}{(1 - 2x^3)}$

$f'(x) = (3x^3+4)(-6x^2) + (9x^2)(1-2x^3)$

$= -18x^5 - 24x^2 + 9x^2 - 18x^5$

$= \boxed{-36x^5 - 15x^2}$

e) $f(t) = \overset{f}{(6 + t^{-2})}\overset{g}{(8t^{10} - 5t^3)}$

$f'(t) = (6 + t^{-2})(80t^9 - 15t^2) + (-2t^{-3})(8t^{10} - 5t^3)$

$= 480t^9 - 90t^2 + 80t^7 - 15 - 16t^7 + 10$

$= \boxed{480t^9 + 64t^7 - 90t^2 - 5}$

f) $f(t) = \overset{f}{(at + b)}\overset{g}{(ct^2 - d)}$

$f'(t) = (at+b)(2ct) + (a)(ct^2 - d)$

$= 2act^2 + 2bct + act^2 - ad$

$= \boxed{3act^2 + 2bct - ad}$

g) $g(u) = \overset{f}{\sqrt{u}}\overset{g}{(2 - u^2 + 5u^4)}$

$g'(u) = \sqrt{u}(-2u + 20u^3) + \frac{1}{2}u^{-\frac{1}{2}}(2 - u^2 + 5u^4)$

$= -2u^{3/2} + 20u^{7/2} + \frac{2 - u^2 + 5u^4}{2u^{\frac{1}{2}}}$

$= -2u^{3/2} + 20u^{7/2} - \frac{1}{2}u^{3/2} + \frac{5}{2}u^{7/2} + \frac{1}{u^{\frac{1}{2}}}$

$= -\frac{5}{2}u^{3/2} + \frac{45}{2}u^{7/2} + \frac{1}{u^{\frac{1}{2}}}$

$= \boxed{45u^{7/2} - \frac{5}{2}u^{3/2} + u^{-\frac{1}{2}}}$

h) $g(v) = \overset{f}{(v - \sqrt{v})}\overset{g}{(v^2 + \sqrt{v})}$

$g'(v) = (v - \sqrt{v})(2v + \frac{1}{2}v^{-\frac{1}{2}}) + (1 - \frac{1}{2}v^{-\frac{1}{2}})(v^2 + \sqrt{v})$

$= 2v^2 + \frac{1}{2}v^{\frac{1}{2}} - 2v^{3/2} - \frac{1}{2} + v^2 + v^{\frac{1}{2}} - \frac{1}{2}v^{3/2} - \frac{1}{2}$

$= \boxed{3v^2 - \frac{5}{2}v^{3/2} + \frac{3}{2}v^{\frac{1}{2}} - 1}$

$fg' + f'g$

3. Find the slope of the tangent to the given curve at the point whose x-coordinate is given.

a) $y = (1-2x)(3x-4); x = 2$

f g

$y' = (1-2x)(3) + (-2)(3x-4)$

$= 3 - 6x - 6x + 8$

$= -12x + 11$

at $x = 2$

$-12(2) + 11$

$-24 + 11$

$\boxed{-13}$

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b) $y = (1 - x + x^2)(x-2); x = 1$

f g

$y' = (1-x+x^2)(1) + (-1+2x)(x-2)$

$= 1 - x + x^2 + (-x + 2 + 2x^2 - 4x)$

$= 1 - x + x^2 - 5x + 2x^2 + 2$

$= 3x^2 - 6x + 3$

at $x = 1$

$3(1)^2 - 6(1) + 3$

$3 - 6 + 3$

$\boxed{0}$

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c) $y = x^4(4x^3+2); x = -1$

f g

$y' = x^4(12x^2) + 4x^3(4x^3+2)$

$= 12x^6 + 16x^6 + 8x^3$

at $x = -1$

$12(-1)^6 + 16(-1)^6 + 8(-1)^3$

$12 + 16 - 8$

$\boxed{20}$

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d) $y = (1 + x - 2x^2)(3x^3 + x - 1); x = 1$

f g

$y' = (1+x-2x^2)(9x^2+1) + (1-4x)(3x^3+x-1)$

$= 9x^2 + 9x^3 - 18x^4 + 1 + x - 2x^2 + 3x^3 + x - 1 - 12x^4 - 4x^2 + 4x$

$= -30x^4 + 12x^3 + 3x^2 + 6x$

at $x = 1$

$-30 + 12 + 3 + 6$

$\boxed{-9}$

e) $y = \overset{f}{x^{-5}}\overset{g}{(1 + x^{-1})}; x = 1$

$$y' = x^{-5}(-x^{-2}) + -5x^{-6}(1+x^{-1})$$

$$= -x^{-7} - 5x^{-6} - 5x^{-7}$$

$$= -6x^{-7} - 5x^{-6}$$

$$\rightarrow \frac{-6}{x^7} - \frac{5}{x^6} \quad \text{at } x=1$$

$$-6-5$$

$$\boxed{-11}$$

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f) $y = \overset{f}{(2 - 3\sqrt{x})}\overset{g}{(4 - \sqrt{x})}; x = 4$

$$y' = (2-3\sqrt{x})\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) + \left(-\frac{3}{2}x^{-\frac{1}{2}}\right)(4-\sqrt{x})$$

$$= -\frac{1}{\sqrt{x}} + \frac{3}{2} - \frac{6}{\sqrt{x}} + \frac{3}{2} \rightarrow -\frac{7}{\sqrt{x}} + 3$$

at $x = 4$

$$-\frac{7}{\sqrt{4}} + 3$$

$$-\frac{7}{2} + \frac{6}{2}$$

$$\boxed{-\frac{1}{2}}$$

4. If $f(x) = (6x^4 - 3x^2 + 1)(2 - x^3)$, find $f'(1)$ by two methods:

a) By using the Product Rule
b) By expanding $f(x)$ first

a) $y' = (6x^4-3x^2+1)(-3x^2) + (24x^3-6x)(2-x^3)$

$$= -18x^6 + 9x^4 - 3x^2 + 48x^3 - 24x^6 - 12x + 6x^4$$

$$= -42x^6 + 15x^4 + 48x^3 - 3x^2 - 12x \quad \text{at } x=1$$

$$= -42+15+48-3-12 = \boxed{6}$$

b) $(6x^4-3x^2+1)(2-x^3)$

$$\rightarrow 12x^4 - 6x^7 - 6x^2 + 3x^5 + 2 - x^3$$

$$y' = 48x^3 - 42x^6 - 12x + 15x^4 - 3x^2$$

at $x=1$

$$= 48-42-12+15-3$$

$$= \boxed{6}$$

5. Find the equation of the tangent line to the curve $y = \overset{f}{(2 - \sqrt{x})}\overset{g}{(1 + \sqrt{x} + 3x)}$ at the point $(1, 5)$.

$$y' = (2-\sqrt{x})\left(\frac{1}{2}x^{-\frac{1}{2}} + 3\right) + -\frac{1}{2}x^{-\frac{1}{2}}(1+\sqrt{x}+3x)$$

$$= \frac{1}{x^{\frac{1}{2}}} + 6 - \frac{1}{2} - 3\sqrt{x} - \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2} - \frac{3\sqrt{x}}{2}$$

at $x=1$

$$\rightarrow 1 + 6 - \frac{1}{2} - 3 - \frac{1}{2} - \frac{1}{2} - \frac{3}{2}$$

$$= \boxed{1}$$

$$y = 1x + b$$

$$5 = 1(1) + b$$

$$4 = b$$

$$\boxed{y = x + 4}$$

6. If $f(2) = 3, f'(2) = 5, g(2) = -1$, and $g'(2) = -4$, find $(fg)'(2)$.

$$(fg)'(2) = f(2)g'(2) + f'(2)g(2)$$
$$= 3(-4) + 5(-1)$$
$$= -12 - 5 = \boxed{-17}$$

7. If f is a differentiable function, find expressions for the derivatives of the following functions.

a) $g(x) = xf(x)$
b) $h(x) = \sqrt{x}f(x)$
c) $F(x) = x^c f(x)$

a) $xf'(x) + f(x)$

b) $\sqrt{x}f'(x) + \frac{1}{2}x^{-\frac{1}{2}}f(x) \rightarrow \sqrt{x}f'(x) + \frac{f(x)}{2\sqrt{x}}$

c) $x^c f'(x) + cx^{c-1}f(x)$

8.

a) Use the Product Rule with $g = f$ to show that if f is differentiable, then

$$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$$

$$[f(x)]^2 = f(x)f(x)$$

$$\frac{d[f(x)]^2}{dx} = f(x)f'(x) + f'(x)f(x) \rightarrow 2f(x)f'(x)$$

b) Use part (a) to differentiate $y = (\underbrace{2 + 5x - x^3}_{f(x)})^2$

$$2[2+5x-x^3][5-3x^2]$$

9.

a) Use the Product Rule twice to show that if $f, g,$ and h are differentiable, then

$$(fgh)' = f'gh + g'fh + h'fg$$

$(fgh)' = f(gh)' + f'gh$

$= f[gh' + g'h] + f'gh$

$= fgh' + fhg' + f'gh \rightarrow f'gh + g'fh + h'fg$

b) Use part (a) to differentiate $y = \sqrt{x}(3x+5)(6x^2-5x+1)$

$f \quad g \quad h$

$\frac{1}{2\sqrt{x}}(3x+5)(6x^2-5x+1) + 3(\sqrt{x})(6x^2-5x+1) + (12x-5)(3x+5)\sqrt{x}$

$\frac{(3x+5)(6x^2-5x+1)}{2\sqrt{x}} + 3\sqrt{x}(6x^2-5x+1) + \sqrt{x}(12x-5)(3x+5)$

10.

a) Taking $f = g = h$ in Question 9, show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

if $(fgh)' = f'gh + g'fh + h'fg$ ad $f = g = h$

then $f'ff + f'ff + f'ff$

$= f'f^2 + f'f^2 + f'f^2 \rightarrow 3f(x)^2 f'(x)$

b) Use part (a) to differentiate $y = (1 + x^3 + x^6)^3$.

$3(x^6 + x^3 + 1)^2(6x^5 + 3x^2)$

11. Use the Principle of Mathematical Induction and the Product Rule to prove the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

When n is a positive integer.

To Prove:

Let $n = 1$

Left Side

$\frac{d}{dx} x = 1$

Right Side

$1(x)^{1-1} = 1$

True for $n=1$

Assume true for $n=k$, so $\frac{d}{dx} x^k = kx^{k-1}$

Let $n = k+1$

Left Side

$\frac{d}{dx} = x^{k+1}$

$= x^k \cdot x$

$= x^k \cdot 1 + kx^{k-1} \cdot x$

$\rightarrow x^k + xkx^{k-1}$

$= x^k + kx^k$

$= x^k(1+k)$

Right Side

$(k+1)x^{(k+1)-1}$

$(k+1)x^k$

LHS = RHS