2.3 The Sum and Difference Rules

The derivative of a sum/difference is the sum/difference of the derivatives

Sum and Difference Rules

If both f and g are differentiable, then so is $f+g$ and $f-g$.

$$(f+g)' = f' + g'$$
$$(f-g)' = f' - g'$$

In Leibniz notation:

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof of the Sum Rule

Let $F = f+g$, that is, $F(x) = (f+g)(x) = f(x)+g(x)$.

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] && \text{(by rearranging terms)} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(by Property 1 of limits)} \\ &\boxed{= f'(x)+g'(x)} \end{aligned}$$

The difference rule can be proved in a similar way.

Ex. 1

Find the derivatives of the following functions.

(a) $f(x) = 2x^4 + \sqrt{x}$

$f(x) = 2x^4 + x^{\frac{1}{2}}$

$f'(x) = 4(2x^3) + \frac{1}{2}x^{-\frac{1}{2}}$

$= \boxed{8x^3 + \frac{1}{2\sqrt{x}}}$

(b) $g(x) = 6x^4 - 5x^3 - 2x + 17$

$g'(x) = 4(6x^3) - 3(5x^2) - 2$

$g'(x) = 24x^3 - 15x^2 - 2$

Ex. 2

Differentiate the following function.

$$y = \left(x - \frac{2}{\sqrt{x}}\right)^2$$

• we will tackle this a new way later!
• for now expand and simplify

$\left(x - \frac{2}{\sqrt{x}}\right)\left(x - \frac{2}{\sqrt{x}}\right)$

$x^2 - \frac{2x}{\sqrt{x}} - \frac{2x}{\sqrt{x}} + \frac{4}{x}$

$x^2 - \frac{4x}{\sqrt{x}} + \frac{4}{x}$

$x^2 - 4\sqrt{x} + 4x^{-1}$

so $y' = 2x - \frac{4}{2}x^{-\frac{1}{2}} - 4x^{-2}$

$$\boxed{y' = 2x - \frac{2}{\sqrt{x}} - \frac{4}{x^2}}$$

Ex. 3

Find the equations of both lines that pass through the point $P(2, 9)$ and are tangent to the parabola $y = 2x - x^2$.

First we need to find the two points on $y = 2x - x^2$ that have the right tangent line slope. Consider point $Q(x, 2x - x^2)$

slope: $\dfrac{9 - (2x - x^2)}{2 - x} = m_{PQ}$

Tangent Line Slope of $y = 2x - x^2$ is $y' = 2 - 2x$

These need to be equal

$$\frac{9 - 2x + x^2}{2 - x} = 2 - 2x \rightarrow 9 - 2x + x^2 = (2 - 2x)(2 - x)$$

$$9 - 2x + x^2 = 4 - 2x - 4x + 2x^2$$

$$0 = -5 - 4x + x^2$$

$$0 = x^2 - 4x - 5$$

$$0 = (x - 5)(x + 1)$$

$$\boxed{x = 5 \quad x = -1}$$

Find the Tangent Line Slope

$x = 5$

$y' = 2 - 2(5)$

$y' = -8$

$x = -1$

$y' = 2 - 2(-1)$

$y' = 4$

if $x = 5$

$y = 2(5) - 5^2$

$y = -15$

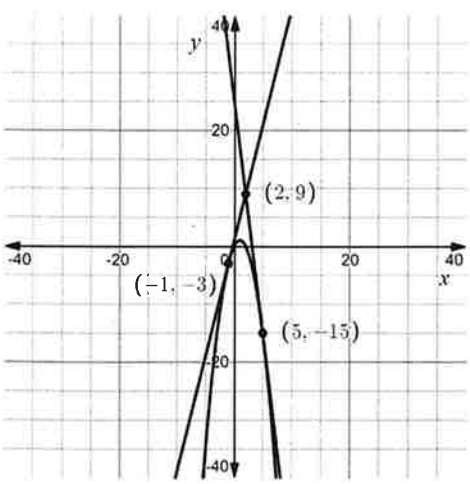
$\boxed{(5, -15)}$

if $x = -1$

$y = 2(-1) - (-1)^2$

$= -3$

$\boxed{(-1, -3)}$



$y = -8x + b$

$-15 = -8(5) + b$

$-15 = -40 + b$

$35 = b$

$\boxed{y = -8x + 35}$

$y = 4x + b$

$-3 = 4(-1) + b$

$-3 = -4 + b$

$1 = b$

$\boxed{y = 4x + 1}$

Homework Assignment

- Exercise 2.3: #1acegik, 2ac, 3 – 7, 9