2.2 The Power Rule

It would be extremely tedious to have to resort to using the definition of derivative every time we wanted to calculate a derivative. We will use the definition of derivative to prove some very simple rules for differentiating polynomials.

Constant Rule
If f is a constant function, $f(x) = c$, then $f'(x) = 0$.
In Leibniz notation:
$$\frac{d}{dx}(c) = 0$$

Proof

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} \\ &= \lim_{h\to 0} 0 \\ &\boxed{= 0} \end{aligned}$$

Ex. 1
Find the derivative of each.

(a) $f(x) = 7$ $\rightarrow f'(x) = 0$

(b) $y = \pi$ $\rightarrow y' = 0$

(c) $\frac{d}{dx}(-4.5)$ $\frac{dy}{dx} = 0$

The next rule allows us to differentiate power functions of the form $f(x) = x^n$.

Power Rule
If $f(x) = x^n$, where n is a positive integer, then
$$f'(x) = nx^{n-1}$$
In Leibniz notation:
$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof
The following equation will be used in this proof when we use the definition of derivative. To prove the rule works for all values of n we must prove the general case.

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

This formula can be easily verified inductively by choosing a value of n and multiplying out the right side of the equation.

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{x^n - a^n}{x-a} \end{aligned}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x-a}$$
$$= \lim_{x \to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$
$$= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1}$$
$$\boxed{= na^{n-1}}$$

Ex. 2

Find the derivative of each.

(a) $f(x) = x^7$

(b) $y = x^{100}$

(c) $y = t^5$

(d) $\frac{d}{du}(u^9)$

a) $f'(x) = 7x^6$

b) $y' = 100x^{99}$

c) $y' = 5t^4$

d) $\frac{dy}{du} = 9u^8$

Ex. 3

Find the equation of the tangent line to the curve $y = x^6$ at the point $(-2, 64)$.

$y' = 6x^5$ at $x = -2$

$y' = 6(-2)^5$

$= -192$ ← Slope of the tangent line at $(-2, 64)$

$y = -192x + b$

$64 = -192(-2) + b$

$64 = 384 + b$

$-320 = b$

$\boxed{y = -192x - 320}$

Although we have proved the Power Rule for exponents n that are positive integers, it turns out to work for any $\boxed{\text{real number } n}$. This fact will be proved later.

± Integers

± Fractions

General Power Rule

If n is any real number, then
$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Ex. 4

Differentiate each of the following.

(a) $f(x) = \frac{1}{x^3}$

x^{-3}

$f'(x) = -3x^{-4}$

$\boxed{f'(x) = \frac{-3}{x^4}}$

(b) $y = \sqrt{x}$

$x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

Rationalize Denominator

$y' = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$

$\boxed{\frac{\sqrt{x}}{2x}}$

How do we differentiate a constant that is multiplied by a function?

Constant Multiple Rule

If $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

In Leibniz notation:
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\
&= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{(by Property 3 of limits)}\\
&\boxed{= cf'(x)}
\end{aligned}
$$

Ex. 5
Differentiate each of the following.

(a) $f(x) = 8x^3$

$8f'(x)x^3$

$8(3x^2)$

$\boxed{24x^2}$

(b) $y = 6x^{\frac{8}{3}}$

$6f'(x)x^{8/3}$

$6\cdot\frac{8}{3}x^{5/3} \rightarrow \boxed{16x^{5/3}}$

Ex. 6
At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

Find $\frac{dy}{dx}$, isolate y

same slope

change to $y=mx+b$

$y=-3x$

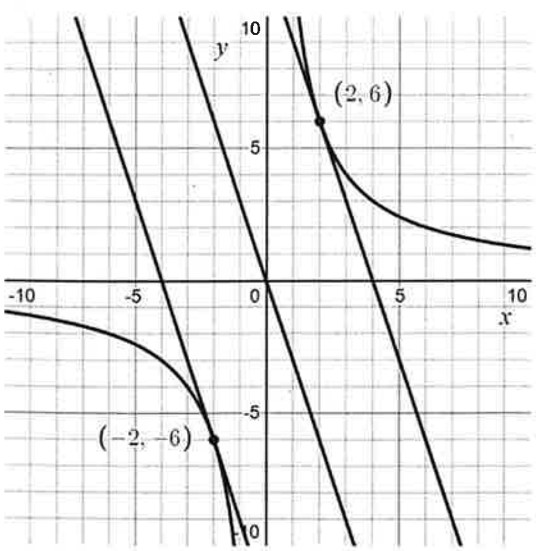
$\boxed{\text{slope is } -3}$

$xy=12$

$y=\frac{12}{x} \rightarrow y=12x^{-1}$

$y'=12(-1)x^{-2} \rightarrow y' \text{ or } \frac{dy}{dx} = -\frac{12}{x^2}$

these need to be the same



$-\frac{12}{x^2} = -3$

$-12=-3x^2$

$4=x^2$

$x=\pm 2$

if $x=2$: $y=\frac{12}{2}$, $y=6$, $(2,6)$

$x=-2$: $y=\frac{12}{-2}$, $y=-6$, $(-2,-6)$

Points are

Ex. 7
A ball is dropped from the upper observation deck of the CN Tower. How fast is the ball falling after 3 s?

Say the top of the tower as position zero with down as negative. The position of the ball as a function of time is:

$v_0 = 0$

$$s(t) = \cancel{v_0 t} + \frac{1}{2}(a)t^2$$

$a = -9.80 \, m/s^2$

$v_0 = 0 \, m/s$ (dropped from rest)

$$s(t) = \frac{1}{2}at^2$$

$$s(t) = \frac{1}{2}(-9.8)t^2 \rightarrow -4.9t^2 = s(t)$$

Derivative of $s(t) = v(t)$

$$s'(t) = v(t)$$

$$s'(t) = -9.8t = v(t)$$

$$v(3) = -9.8(3)$$

$$= \boxed{-29.4 \, m/s}$$

Homework Assignment
Exercise 2.2: #1, 2, 3ace, 4, 5, 7, 8