Exercise 2.2 – Practice Problems

1. State the derivative of each function.

a) $f(x) = 32$

$f'(x) = 0$

b) $f(x) = x^4$

$f'(x) = 4x^3$

c) $y = x^{12}$

$f'(x) = 12x^{11}$

d) $y = -3.724$

$y' = 0$

e) $f(x) = x$

$f'(x) = 1$

f) $f(x) = x^{\pi}$

$f'(x) = \pi x^{\pi - 1}$

g) $f(x) = x^{43}$

$f'(x) = 43x^{42}$

h) $f(x) = 2^5$

$f'(x) = 0$

i) $g(x) = x^{-2}$

$g'(x) = -2x^{-3} \rightarrow -\frac{2}{x^3}$

j) $g(x) = x^{\frac{3}{2}}$

$g'(x) = \frac{3}{2}x^{\frac{1}{2}}$

2. Differentiate.

a) $f(x) = 8x^{12}$

$12 \cdot 8x^{11}$

$96x^{11}$

b) $f(x) = -3x^9$

$-27x^8$

c) $f(t) = 3t^{\frac{4}{3}}$

$\frac{4}{3} \cdot 3t^{\frac{1}{3}} \rightarrow 4t^{\frac{1}{3}}$

d) $g(t) = 8t^{-\frac{3}{4}}$

$-\frac{3}{4}(8)t^{-7/4} \rightarrow -6t^{-7/4}$

e) $y = \frac{1}{x^4}$

$y = x^{-4} \rightarrow y' = -4x^{-5}$

$y' = -\frac{4}{x^5}$

f) $y = \frac{2}{x^2} \rightarrow 2x^{-2}$

$y' = -4x^{-3} = -\frac{4}{x^3}$

g) $g(t) = (2t)^3$ $= 8t^3$

$8 \cdot 3t^2 \rightarrow 24t^2$

h) $h(y) = \left(\frac{y}{3}\right)^2$

$\frac{y^2}{9} \rightarrow \frac{1}{9} \cdot 2y \rightarrow \frac{2}{9}y$

i) $f(x) = \sqrt[3]{x}$

$x^{\frac{1}{3}} \rightarrow \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

j) $f(x) = \sqrt[3]{x^2}$

$x^{2/3} \rightarrow \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$

k) $y = \frac{1}{\sqrt{x}}$

$x^{-\frac{1}{2}} \rightarrow -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$

l) $y = \frac{3}{\sqrt[4]{x}}$

$3x^{-\frac{1}{4}} \rightarrow 3 \cdot -\frac{1}{4}x^{-5/4} = \frac{-3}{4x^{5/4}}$

m) $y = \sqrt{3}x^{\sqrt{2}}$

$\sqrt{3} \cdot \sqrt{2}x^{\sqrt{2}-1} = \sqrt{6}x^{\sqrt{2}-1}$

n) $y = (x^3)^4$

$x^{12} \rightarrow 12x^{11}$

3. Find the slope of the tangent line to the graph of the given function at the point whose x-coordinate is given.

a) $f(x) = 2x^3; x = \frac{1}{3}$

$2 \cdot 3x^2$

$6x^2$ at $x = \frac{1}{3}$ $\rightarrow 6\left(\frac{1}{3}\right)^2$

$\frac{6}{9} = \boxed{\frac{2}{3}}$

b) $f(x) = x^{1.4}; x = 1$

$1.4x^{.4}$

$1.4(1)^{.4} = \boxed{1.4}$

c) $g(x) = x^{-3}; x = -1$

$-3x^{-4} \rightarrow \frac{-3}{(-1)^4} = \boxed{-3}$

d) $g(x) = \sqrt[5]{x}; x = 32$

$x^{\frac{1}{5}} \rightarrow \frac{1}{5}x^{-4/5} \rightarrow \frac{1}{5x^{4/5}} = \frac{1}{5 \cdot 32^{4/5}} = \frac{1}{5 \cdot 2^4}$

$\boxed{\frac{1}{80}}$

e) $y = \sqrt{x^3}; x = 8$

$x^{3/2} \rightarrow \frac{3}{2}x^{+\frac{1}{2}} = \frac{3(8)^{\frac{1}{2}}}{2}$

$\rightarrow \frac{3\sqrt{8}}{2} = \frac{3 \cdot 2\sqrt{2}}{2} = \boxed{3\sqrt{2}}$

f) $y = \frac{6}{x}; x = -3$

$6x^{-1} \rightarrow -6x^{-2}$

$= -\frac{6}{x^2} \rightarrow \frac{-6}{(-3)^2} = -\frac{6}{9} = \boxed{-\frac{2}{3}}$

4. Find the equation of the tangent line to the curve at the given point.

a) $y = x^5$; $(2, 32)$

$y' = 5x^4$

at $x = 2$

$y' = 5(2)^4 = 80$

$y = 80x + b$

$32 = 80(2) + b$

$-128 = b$

$\boxed{y = 80x - 128}$

b) $y = 2\sqrt{x}$; $(9, 6)$

$2x^{\frac{1}{2}}$

$y' = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}}$

$y' = \frac{1}{x^{\frac{1}{2}}}$

at $x = 9$

$y' = \frac{1}{3}$

$y = \frac{1}{3}x + b$

$6 = \frac{1}{3}(9) + b \quad b = 3$

$\boxed{y = \frac{1}{3}x + 3}$

c) $xy = 1$; $\left(5, \frac{1}{5}\right)$

$y = \frac{1}{x}$

$y = x^{-1}$

$y' = -x^{-2}$

$y' = -\frac{1}{x^2}$

at $x = 5 \quad y' = \frac{-1}{25}$

$y = \frac{-1}{25}x + b$

$\frac{1}{5} = -\frac{1}{25}(5) + b \quad b = \frac{2}{5}$

$\boxed{y = \frac{-1}{25}x + \frac{2}{5}}$

d) $y = \sqrt[3]{x}$; $(-8, -2)$

$y = x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{-2/3}$

$y' = \frac{1}{3x^{2/3}}$

at $x = -8$

$\frac{1}{3(-8)^{2/3}} = \frac{1}{12}$

$y = \frac{1}{12}x + b$

$-2 = \frac{1}{12}(-8) + b$

$-2 = -\frac{2}{3} + b \quad b = -\frac{4}{3}$

$\boxed{y = \frac{1}{12}x - \frac{4}{3}}$

5. Use the definition of derivative to show that

If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$

(This proves the Power Rule for the case $n = -1$.)

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$\lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$

$\lim_{h \to 0} \frac{-h}{hx(x+h)}$

$\lim_{h \to 0} \frac{-1}{x(x)} = -\frac{1}{x^2}$

6. Use the definition of derivative to show that

If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$

(This proves the Power Rule for the case $n = 1/2$.)

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$

$\frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$

$\lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

$\frac{1}{\sqrt{x} + \sqrt{x}}$

$\boxed{\frac{1}{2\sqrt{x}}}$

7. At what point on the parabola $y = 3x^2$ is the slope of the tangent line equal to 24?

$y' = 6x$

$6x = 24$

$x = 4$

$y = 3(4)^2$

$y = 3(16)$

$y = 48$

at the point $(4, 48)$

8. Find the point on the curve $y = x\sqrt{x}$ where the tangent line is parallel to the line $6x - y = 4$?

$y = x \cdot x^{\frac{1}{2}}$

$y = x^{3/2}$

$y' = \frac{3}{2}x^{\frac{1}{2}}$

$6 = \frac{3}{2}x^{\frac{1}{2}}$

$12 = 3x^{\frac{1}{2}}$

$4 = x^{\frac{1}{2}}$

$x = 16$

$y = 16\sqrt{16}$

$= 16(4)$

$= 64$

$(16, 64)$

$-y = -6x + 4$

$y = 6x - 4$

$m = 6$

9. At what point on the curve $y = -2x^4$ is the tangent line perpendicular to the line $x - y + 1 = 0$?

$y' = -8x^3$

$-8x^3 = -1$

$x^3 = \frac{1}{8}$

$x = \frac{1}{2}$

$y = -2\left(\frac{1}{2}\right)^4$

$= -2\left(\frac{1}{16}\right) = -\frac{1}{8}$

negative reciprocal

$y = x + 1$

need $m = -1$

$\left(\frac{1}{2}, -\frac{1}{8}\right)$

10. Find the points on the curve $y = 1 - \frac{1}{x}$ where the tangent line is perpendicular to the line $y = 1 - 4x$.

need $m = \frac{1}{4}$

$y = -x^{-1} + 1$

$y' = x^{-2}$

$y' = \frac{1}{x^2}$

$\frac{1}{x^2} = \frac{1}{4}$

$x = \pm 2$

if $x = 2$

$y = 1 - \frac{1}{2}$

$y = \frac{1}{2}$

$\left(2, \frac{1}{2}\right)$

$x = -2$

$y = 1 - \frac{1}{-2}$

$y = 3/2$

$\left(-2, 3/2\right)$

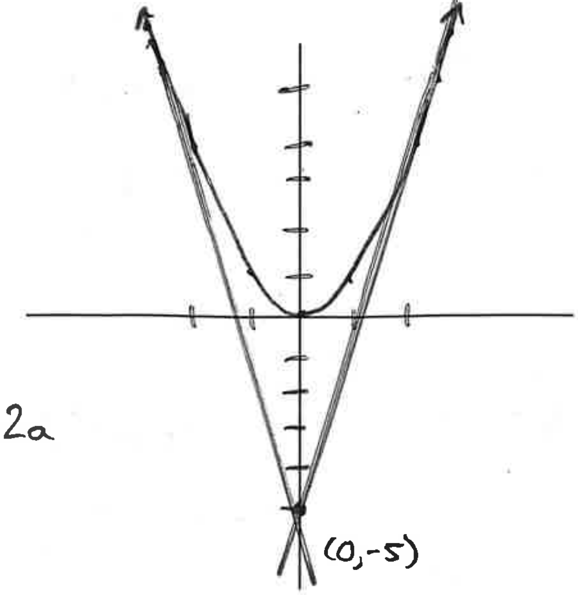
11. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -5)$. Find the coordinates of the points where these tangent lines meet the parabola.

$y = x^2$

$y' = 2x$

Let the x-coordinate be a.

we have a point at (a, a^2)



Slope of tangent line give by

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\dfrac{a^2 - (-5)}{a - 0}$

$m = \dfrac{a^2 + 5}{a}$

$\dfrac{a^2+5}{a} = 2x = 2a$

$\dfrac{a^2+5}{a} = 2a$

$a^2 + 5 = 2a^2$

$5 = a^2$

$a = \pm\sqrt{5}$

if $a = \sqrt{5}$ $\quad y = 5$

$a = -\sqrt{5}$ $\quad y = 5$

12. A manufacturer of cartridges for stereo systems has designed a stylus with a parabolic cross-section as shown in the image below. The equation of the parabola is $y = 8x^2$, where x and y are measured in millimetres. If the stylus sits in a record groove whose sides make an angle of θ with the horizontal direction, where $\tan\theta = 2.5$, find the points of contact P and Q of the stylus with the groove.

$y = 8x^2$

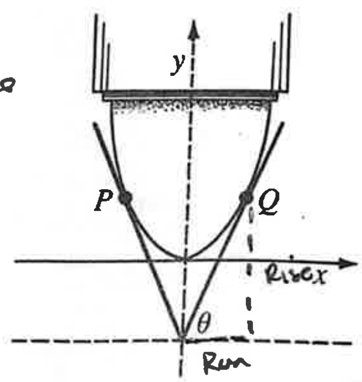
$\tan\theta = 2.5$

$\tan\theta = \dfrac{\text{RISE}}{\text{Run}}$

slope of tangent line is $\tan\theta$

slopes are:

2.5 and -2.5



$y' = 16x$

so $16x = 2.5 \rightarrow x = \dfrac{5/2}{16} \rightarrow \dfrac{5}{32}$ $\quad y = 8\left(\dfrac{5}{32}\right)^2 = \dfrac{50}{256}$ $\quad \left(\dfrac{5}{32}, \dfrac{50}{256}\right)$

$16x = -2.5$ $\quad x = \dfrac{-5/2}{16}$ $\quad -\dfrac{5}{32}$ $\quad y = 8\left(\dfrac{-5}{32}\right)^2 = \dfrac{50}{256}$ $\quad \left(-\dfrac{5}{32}, \dfrac{50}{256}\right)$