2.1 Derivatives

A **derivative** is defined as a limit and is used to calculate the slope of the line tangent to a curve. From section 1.4 the slope of the tangent line was defined as

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

And in section 1.5 it was used to calculate the instantaneous velocity from the position function for an object. In fact, any instantaneous rate of change can be calculated in this way. Therefore, derivatives have important applications in all branches of science and engineering.

The **derivative of a function f at a number a** is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If the limit exists.

The derivative is the same thing!

Another way to define the derivative of a function at a point a is

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Ex. 1
If $f(x)=2x^2-5x+6$, find $f'(4)$, the derivative of f at 4.

$$f'(4)=\lim_{h\to 0}\frac{2(4+h)^2-5(4+h)+6-[2(4)^2-5(4)+6]}{h}$$

$$\lim_{h\to 0}\frac{2(16+8h+h^2)-20-5h+6-32+20-6}{h}$$

$$\lim_{h\to 0}\frac{\cancel{32}+16h+2h^2-\cancel{20}-5h\cancel{+6}-\cancel{32}+\cancel{20}\cancel{-6}}{h}$$

$$\lim_{h\to 0}\frac{16h+2h^2-5h}{h}=2h+11=11$$

$f'(4)=11$

Slope of tangent line at $x=4$ is 11

the derivative at $f'(4)$ is 11.

Interpretation of the Derivative	
1.	**As the slope of the tangent.** The tangent line to the curve $y=f(x)$ at the point $(a,f(a))$ is the line through $(a,f(a))$ with slope $f'(a)$.
2.	**As a rate of change.** The **instantaneous** rate of change of $y=f(x)$ with respect to x when $x=a$ is equal to $f'(a)$. For a moving object, if $s=f(t)$ is the position function of an object then $v=f'(a)$ is the instantaneous velocity of the object at time $t=a$.

$$\frac{f(a+h)-f(a)}{h}$$

Ex. 2
Find the derivative of $f(x) = x^2 - 3x$ at any number a. Then use it to find the slopes of the tangents to $f(x)$ when $x = 1, 2, 3, 4$.

Use the equation and solve in the abstract

$$f'(a) \lim_{h\to 0} \frac{(a+h)^2 - 3(a+h) - [a^2 - 3a]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{3a} - 3h - \cancel{a^2} + \cancel{3a}}{h}$$

$$\lim_{h\to 0} \frac{2ah + h^2 - 3h}{h} = \boxed{2a-3} = f'(a)$$

Equation of the slope of the tangent line (the derivative) as a changes.

$f'(1) = 2(1) - 3 = -1$

$f'(2) = 2(2) - 3 = 1$

$f'(3) = 2(3) - 3 = 3$

$f'(4) = 2(4) - 3 = 5$

The Derivative as a Function

As can be seen from the previous example, the derivative of a function $f(x)$ can be calculated at any value of x that is part of the domain of the derivative $f'(x)$ by allowing the x to vary throughout the domain of $f'(x)$. In this way, the derivative of a function can be thought of as another function onto itself.

> Given a function f, the **derivative** of f is the function f' defined by
>
> $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of this new function f' is the set of all numbers x for which the limit exists. Since the definition of $f'(x)$ contains the original function $f(x)$, the domain of f' will always be a **subset** (or possibly the same domain) as f.

Ex. 3
Find the derivative of the function $f(x) = x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \rightarrow \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \rightarrow \frac{2xh + h^2}{h} \rightarrow \lim_{h\to 0} 2x + h = \boxed{2x}$$

The derivative of the function $f(x) = x^2$ is $\boxed{f'(x) = 2x}$

Ex. 4
If $f(x) = \sqrt{x+2}$, find f' and state the domains of f and f'.

conjugate!

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \to \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{(\sqrt{x+h+2}+\sqrt{x+2})}{(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$\lim_{h\to 0} \frac{x+h+2-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} \to \frac{\cancel{x}+h+\cancel{2}-\cancel{x}-\cancel{2}}{h(\sqrt{x+h+2}+\sqrt{x+2})} \to \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$\lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{\sqrt{x+2}+\sqrt{x+2}} = \boxed{\frac{1}{2\sqrt{x+2}} = f'(x)}$$

Ex. 5
Find f' if $f(x) = \frac{x+1}{3x-2}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h+1)}{3(x+h)-2} - \left[\frac{x+1}{3x-2}\right]}{h}$$

$$\lim_{h\to 0} \frac{\frac{(x+h+1)}{3x+3h-2} - \frac{(x+1)}{(3x-2)}}{h} \to \frac{\frac{(3x-2)(x+h+1)-(x+1)(3x+3h-2)}{(3x+3h-2)(3x-2)}}{h}$$

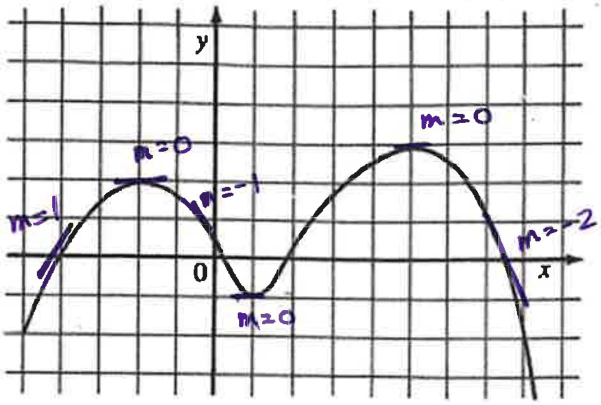
$$\lim_{h\to 0} \frac{3x^2+3xh+3x-2x-2h-2-[3x^2+3xh-2x+3x+3h-2]}{h(3x+3h-2)(3x-2)}$$

$$\lim_{h\to 0} \frac{\cancel{3x^2}+\cancel{3xh}+\cancel{x}-2h-\cancel{2}-\cancel{3x^2}-\cancel{3xh}-\cancel{x}-3h+\cancel{2}}{h(3x+3h-2)(3x-2)} = \frac{-5\cancel{h}}{\cancel{h}(3x+3h-2)(3x-2)}$$

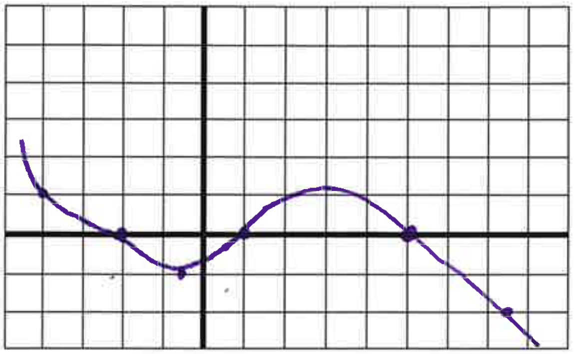
$$\lim_{h\to 0} \frac{-5}{(3x+3h-2)(3x-2)} \to \frac{-5}{(3x-2)(3x-2)} = \boxed{-\frac{5}{(3x-2)^2} = f'(x)}$$

Ex. 6

Use the given graph of f to sketch the graph of f'



The slope at given x-values in f(x) translate to the x-values in f'(x)



Graph of f'(x)

Other Notations

German mathematician Gottfried Leibniz developed another notation for writing derivatives in the mid-1670s:

$$\text{If } y = f(x)\text{, we write } \frac{dy}{dx} = f'(x)$$

The results of Examples 3, 4, and 5 are expressed as follows.

$$\text{If } y = x^2\text{, then } \frac{dy}{dx} = 2x$$

$$\text{If } y = \sqrt{x+2}\text{, then } \frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}$$

$$\text{If } y = \frac{x+1}{3x-2}\text{, then } \frac{dy}{dx} = -\frac{5}{(3x-2)^2}$$

This notation can serve as a reminder of the procedure for finding a derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

horizontal axis

vertical axis

This is because both the independent variable x and the dependent variable y are indicated. For instance, if the position s of a particle is given as a function of time t, then the instantaneous velocity of the particle is expressed as

$$v = \frac{ds}{dt}$$

same as $s'(t) = v(t)$

Using the Leibniz notation, we can think of the process of finding the derivative of a function as an operation called **differentiation**, which is performed on a function f to produce a new function f' called the derivative. Thought of this way we can write

$y = f(x)$

not inverse but derivative

$$\frac{dy}{dx} = \frac{d}{dx} f(x)$$

In this way, $\frac{d}{dx}$ can be thought of as a differentiation operator. So, we could write

$y = x^2 \to \frac{d}{dx}(y) = \frac{d}{dx}(x^2)$

$$\frac{d}{dx}(x^2) = 2x \quad \text{and} \quad \frac{d}{dx}\sqrt{x+2} = \frac{1}{2\sqrt{x+2}}$$

Sometimes the symbols D or D_x (meaning differentiate with respect to x) are also used as differentiation operators. Thus, we have the following equivalent notations for the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x y$$

In Leibniz notation, if you want to indicate that the derivative should be evaluated at a specific number a, write:

evaluate the derivative at $x = a$

$$\left.\frac{dy}{dx}\right]_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=a}$$

Which is the same as $f'(a)$.

Just because you can evaluate $f(a)$ doesn't mean you can evaluate $f'(a)$

Differentiable Functions

* A function f is **differentiable** at a if $f'(a)$ exists. A function is **differentiable on an interval** if it is differentiable at every number in that interval. In **Ex. 3** $f(x) = x^2$ is differentiable on $\mathbb{R}$ and in **Ex. 4** $f(x) = \sqrt{x+2}$ is differentiable for $\boxed{x > -2.}$

Domain of $f'(x)$

$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Ex. 7
Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

Does this limit exist

Check two sided limits

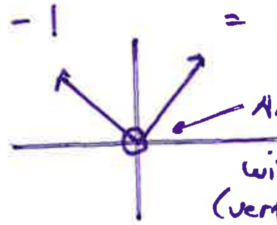
$\lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$

$\lim_{h \to 0^-} \frac{-h}{h} = -1$ $\quad$ $\lim_{h \to 0^+} \frac{h}{h} = 1$ $\quad$ since LSL ≠ RSL

Limit Does Not Exist

Homework Assignment

- Exercise 2.1: #1 – 6, 8 – 13



Any function with a sharp corner (vertex) can't be differentiable at that point