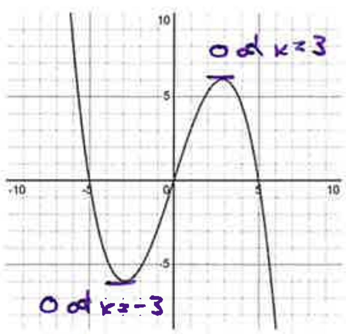
Exercise 2.1 – Practice Problems

1. Each of the following limits represents the derivative of some function f at some number a. State f and a in each case. * A number of potential solutions *

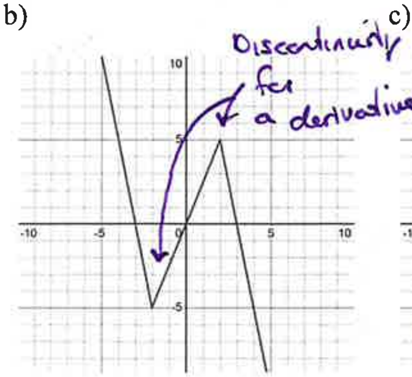
a) $\lim_{h\to 0} \frac{(3+h)^2-3^2}{h}$ $a=3$ $f(x)=x^2$	b) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$ $a=2$ $f(x)=x^3$
c) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$ $a=4$ $f(x)=\sqrt{x}$	d) $\lim_{h\to 0} \frac{\left[(1+h)^4+3(1+h)\right]-4}{h}$ $a=1$ $f(x)=x^4+3x$
e) $\lim_{h\to 0} \frac{2^{1+h}-2}{h}$ $a=1$ $f(x)=2^x$	f) $\lim_{x\to 1} \frac{x^5-1}{x-1}$ $\leftarrow \frac{f(x)-f(a)}{x-a}$ $a=1$ $f(x)=x^5$

2. The graph of f is given. Match it with the graph of its derivative.

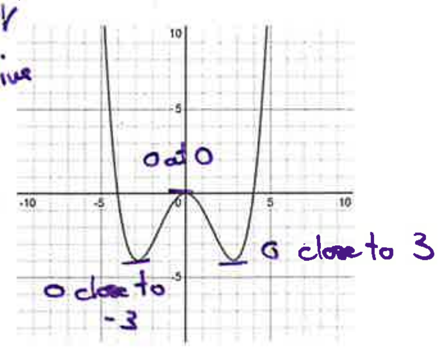
a)



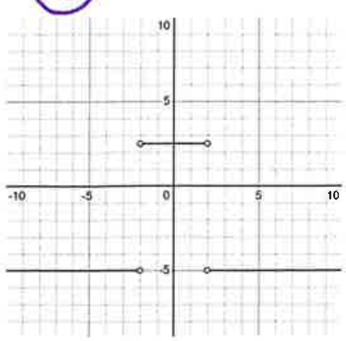
b)



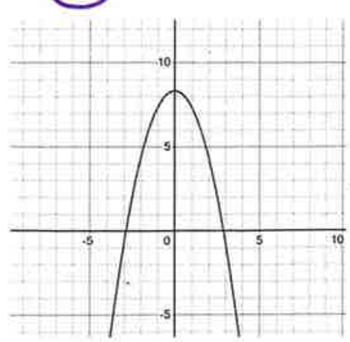
c)



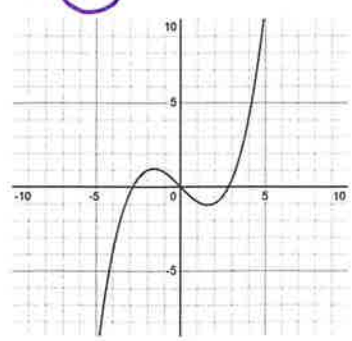
i) b



ii) a

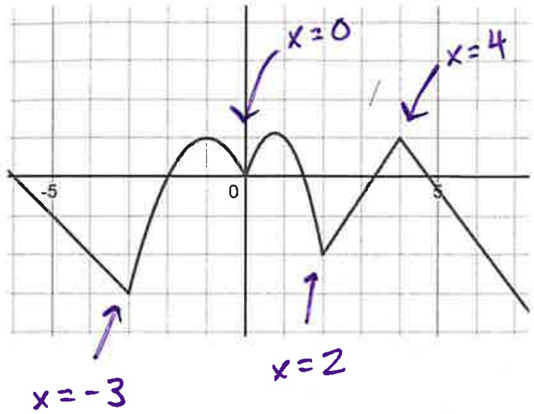


iii) c

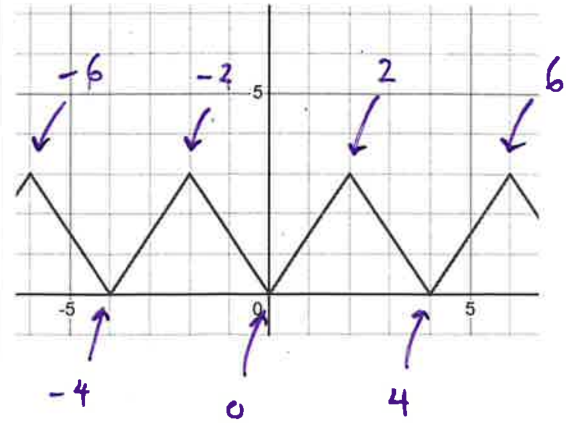


3. At what values of x are the functions not differentiable?

a)



b)



No sharp points

4. If $f(x) = x^2 + 7x$, find $f'(3)$. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ $a = 3$

Limit Equation

$$f'(3) = \lim_{h\to 0} \frac{(3+h)^2 + 7(3+h) - [3^2 + 7(3)]}{h} \rightarrow \frac{13h + h^2}{h} = 13 + h$$

$$f'(3) = \lim_{h\to 0} \frac{9 + 6h + h^2 + 21 + 7h - 30}{h} \qquad = \boxed{13}$$

5. If $g(x) = 15 - 3x^2$, find $g'(-1)$.

$$g'(-1) = \lim_{h\to 0} \frac{15 - 3(-1+h)^2 - [15 - 3(-1)^2]}{h}$$

$$\lim_{h\to 0} \frac{15 - 3(1 - 2h + h^2) - 12}{h} \rightarrow \lim_{h\to 0} \frac{15 - 3 + 6h - 3h^2 - 12}{h}$$

$$\lim_{h\to 0} \frac{6h - 3h^2}{h} = 6 - 3h = \boxed{6}$$

6. If $f(x) = \frac{1}{x}$, find $f'(3)$ and use it to find the equation of the tangent line to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

$$f'(3) = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$\lim_{h\to 0} \frac{\frac{3 - 3 - h}{3(3+h)}}{h}$$

$$\lim_{h\to 0} \frac{-h}{h(9+3h)} \rightarrow \lim_{h\to 0} \frac{-1}{9+3h} = \boxed{-\frac{1}{9}}$$

$$\boxed{y = -\frac{1}{9}x + \frac{2}{3}}$$

$y = -\frac{1}{9}x + b$ at $\left(3, \frac{1}{3}\right)$

$\frac{1}{3} = -\frac{1}{9}(3) + b \rightarrow \frac{1}{3} = -\frac{1}{3} + b$

$b = \frac{2}{3}$

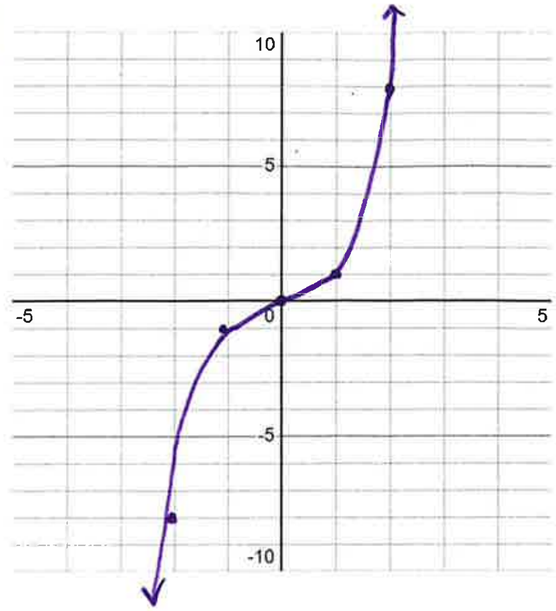
7. If $f(x) = x^3$, find $f'(a)$ and use it to find the slopes of the tangent line to the cubic curve $y = x^3$ at the points $(-1,-1)$, $(0,0)$, $(1,1)$, and $(2,8)$. Illustrate by sketching the curve and these tangents.

$$f'(a) = \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} \rightarrow \frac{(a+h)(a^2+2ah+h^2)-a^3}{h}$$

$$\lim_{h\to 0} \frac{\cancel{a^3}+2a^2h+ah^2+a^2h+2ah^2+h^3-\cancel{a^3}}{h}$$

$$\lim_{h\to 0} \frac{3a^2h+3ah^2+h^3}{h} = 3a^2+3ah+h^2$$

$$= \boxed{3a^2}$$



at $a=-1$ $m=3$ $y=3x+b \rightarrow y=3x+2$

$a=0$ $m=0$ $y=0x+b \rightarrow y=0$

$a=1$ $m=3$ $y=3x+b \rightarrow y=3x-2$

$a=2$ $m=12$ $y=12x+b \rightarrow y=12x-16$

$$(a+h)^3 = a^3+3a^2h+3ah^2+h^3$$

8. Find $f'(a)$ for each of the following functions.

a) $f(x) = 7x - x^2$

$$f'(a) = \lim_{h\to 0} \frac{7(a+h)-(a+h)^2-[7a-a^2]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{7a}+7h-\cancel{a^2}-2ah-h^2-\cancel{7a}+\cancel{a^2}}{h}$$

$$\lim_{h\to 0} \frac{7h-2ah-h^2}{h} = 7-2a-h$$

$$= \boxed{7-2a}$$

b) $f(x) = 2x^3 + 5$

$$f'(a) = \lim_{h\to 0} \frac{2(a+h)^3+5-[2a^3+5]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{2a^3}+6a^2h+6ah^2+2h^3+\cancel{5}-\cancel{2a^3}-\cancel{5}}{h}$$

$$\lim_{h\to 0} \frac{6a^2h+6ah^2+2h^3}{h} = 6a^2+6ah+2h^2$$

$$= \boxed{6a^2}$$

c) $f(x) = \dfrac{1+2x}{1+x}$

$$f'(a) = \lim_{h\to 0} \frac{\frac{1+2(a+h)}{1+a+h} - \frac{1+2a}{1+a}}{h}$$

$$f'(a) = \lim_{h\to 0} \frac{(1+2a+2h)(1+a)-(1+2a)(1+a+h)}{h(1+a+h)(1+a)}$$

$$f'(a) = \lim_{h\to 0} \frac{1+2a+2h+a+2a^2+2ah-[1+a+h+2a+2a^2+2ah]}{h(1+a+h)(1+a)}$$

$$f'(a) \lim_{h\to 0} \frac{\cancel{1}+\cancel{2a}+2h+\cancel{a}+\cancel{2a^2}+\cancel{2ah}-\cancel{1}-\cancel{a}-h-\cancel{2a}-\cancel{2a^2}-\cancel{2ah}}{h(1+a+h)(1+a)}$$

$$f'(a) \lim_{h\to 0} \frac{h}{h(1+a+h)(1+a)} = \frac{1}{(1+a)(1+a)} = \boxed{\frac{1}{(1+a)^2}}$$

d) $f(x) = \sqrt{x}$

$$f'(a) = \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$$

$$f'(a) = \lim_{h\to 0} \frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{a}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

9. The position function of a particle moving along a line is given by $s = f(t) = 5t^2 - 2t + 6$, where t is measured in seconds and s in meters. Find $f'(a)$ and use it to find the velocity of the particle after 1s, 2s, and 3s.

$$f'(a) = \lim_{h\to 0} \frac{5(a+h)^2 - 2(a+h) + 6 - [5a^2 - 2a + 6]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{5a^2} + 10ah + 5h^2 - \cancel{2a} - 2h + \cancel{6} - \cancel{5a^2} + \cancel{2a} - \cancel{6}}{h}$$

$$\lim_{h\to 0} \frac{10ah + 5h^2 - 2h}{h} \rightarrow \boxed{10a - 2}$$

$$\lim_{h\to 0} = 10a + 5h - 2$$

at $a = 1 \rightarrow \boxed{8 m/s}$

$a = 2 \rightarrow \boxed{18 m/s}$

$a = 3 \rightarrow \boxed{28 m/s}$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

10. Find the derivative $f'(x)$ of each function.

a) $f(x) = 3x^2 + 2x - 4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{3(x+h)^2 + 2(x+h) - 4 - [3x^2 + 2x - 4]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{2x} + 2h - \cancel{4} - \cancel{3x^2} - \cancel{2x} + \cancel{4}}{h}$$

$$\lim_{h\to 0} \frac{6xh + 3h^2 + 2h}{h} = \boxed{6x + 2}$$

b) $f(x) = x^2 - x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - (x+h)^3 - [x^2 - x^3]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^3} - 3x^2h - 3xh^2 - h^3 - \cancel{x^2} + \cancel{x^3}}{h}$$

$$\lim_{h\to 0} \frac{2xh + h^2 - 3x^2h - 3xh^2 - h^3}{h}$$

$$\lim_{h\to 0} 2x + h - 3x^2 - 3xh - h^2 = \boxed{2x - 3x^2}$$

c) $f(x) = x^4$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$

$$\lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$\lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$\lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3$$

$$= \boxed{4x^3}$$

d) $f(x) = \dfrac{x}{5x - 1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{x+h}{5(x+h)-1} - \frac{x}{5x-1}}{h}$$

$$\frac{(5x-1)(x+h) - x(5x+5h-1)}{h(5x+5h-1)(5x-1)}$$

$$\frac{\cancel{5x^2} + \cancel{5xh} - \cancel{x} - h - \cancel{5x^2} - \cancel{5xh} + \cancel{x}}{h(5x+5h-1)(5x-1)}$$

$$\lim_{h\to 0} \frac{-h}{h(5x+5h-1)(5x-1)} = \boxed{\frac{-1}{(5x-1)^2}}$$

$$(x+h)^4 = (x+h)(x^3 + 3x^2h + 3xh^2 + h^3) \rightarrow x^4 + 3x^3h + 3x^2h^2 + 3xh^3 + x^3h + 3x^2h^2 + 3xh^3 + h^4$$

$$x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

11. Find the derivative of each function. Find the domains of both the function and its derivative.

a) $f(x) = \sqrt{2x-1}$

Domains
$f(x)$ $x \geq \frac{1}{2}$
$f'(x)$ $x > \frac{1}{2}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$$

$$f'(x) = \frac{(\sqrt{2x+2h-1} - \sqrt{2x-1})(\sqrt{2x+2h-1} + \sqrt{2x-1})}{h(\sqrt{2x+2h-1} + \sqrt{2x-1})}$$

$$\boxed{\frac{1}{\sqrt{2x-1}}} = f'(x) = \frac{2x+2h-1-2x+1}{h(\sqrt{2x+2h-1}+\sqrt{2x-1})} \to \frac{\cancel{2}\,1}{\cancel{2}\sqrt{2x-1}}$$

b) $g(x) = \frac{1}{\sqrt{x}}$

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x+h})(\sqrt{x})} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$\lim_{h\to 0} \frac{x-(x+h)}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})} = \boxed{\frac{-1}{x(2\sqrt{x})}}$$

Domain: $f(x) = x > 0$ $f'(x) = x > 0$

c) $F(x) = \frac{3-2x}{4+x}$

Domain
$f(x): x \neq -4$
$f'(x): x \neq -4$

$$F'(x) = \lim_{h\to 0} \frac{\frac{3-2(x+h)}{4+x+h} - \frac{3-2x}{4+x}}{h}$$

$$F'(x) \lim_{h\to 0} \frac{(3-2x-2h)(4+x) - (3-2x)(4+x+h)}{h(4+x+h)(4+x)}$$

$$F'(x) \lim_{h\to 0} \frac{\cancel{12}+\cancel{3x}-\cancel{8x}-\cancel{2x^2}-8h-\cancel{2xh}-[\cancel{12}+\cancel{3x}+3h-\cancel{8x}-\cancel{2x^2}-\cancel{2xh}]}{h(4+x+h)(4+x)}$$

$$= \frac{-8h}{h(4+x+h)(4+x)} = \boxed{\frac{-8}{(4+x)^2}}$$

d) $f(t) = \frac{2}{t^2-1}$

Domain
$f(t)$ $t \neq \pm 1$
$f'(t)$ $t \neq \pm 1$

$$f'(t) = \frac{\frac{2}{(t+h)^2-1} - \frac{2}{t^2-1}}{h}$$

$$f'(t) = \frac{2(t^2-1) - 2[(t+h)^2-1]}{h(t^2-1)[(t+h)^2-1]}$$

$$f'(t) = \frac{2t^2-2-2t^2-4th-2h^2+2}{h(t^2-1)[(t+h)^2-1]}$$

$$\lim_{h\to 0} \frac{-4th}{h(t^2-1)[(t+h)^2-1]} = \frac{-4}{(t^2-1)(t^2-1)} = \boxed{\frac{-4t}{(t^2-1)^2}}$$

12. Find the derivative $\frac{dy}{dx}$.

a) $y = 7 - 3x$

$$y' = \lim_{h\to 0} \frac{7-3(x+h)-[7-3x]}{h}$$

$$\lim_{h\to 0} \frac{7-3x-3h-7+3x}{h} = \frac{-3h}{h} = \boxed{-3}$$

b) $y = 3x^3 + 2x$

$$y' = \lim_{h\to 0} \frac{3(x+h)^3 + 2(x+h) - [3x^3+2x]}{h}$$

$$\lim_{h\to 0} \frac{3(x^3+3x^2h+3xh^2+h^3)+2x+2h-3x^3-2x}{h}$$

$$\lim_{h\to 0} \frac{\cancel{3x^3}+9x^2h+9xh^2+h^3+\cancel{2x}+2h-\cancel{3x^3}-\cancel{2x}}{h}$$

$$\lim_{h\to 0} \frac{9x^2h+9xh^2+h^3+2h}{h} = \boxed{9x^2+2}$$

c) $y = x + \frac{1}{x} = \frac{x^2+1}{x}$

$$y' = \lim_{h\to 0} \frac{\frac{(x+h)^2+1}{(x+h)} - \frac{(x^2+1)}{x}}{h}$$

$$\lim_{h\to 0} \frac{x[x^2+2xh+h^2+1] - (x^2+1)(x+h)}{h(x+h)x}$$

$$\lim_{h\to 0} \frac{\cancel{x^3}+2x^2h+xh^2+\cancel{x}-\cancel{x^3}-x^2h-\cancel{x}-h}{h(x+h)x}$$

$$\lim_{h\to 0} \frac{x^2h+xh^2-h}{h(x+h)x} = \boxed{\frac{x^2-1}{x^2}}$$

d) $y = \frac{1}{x^2}$

$$y' = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

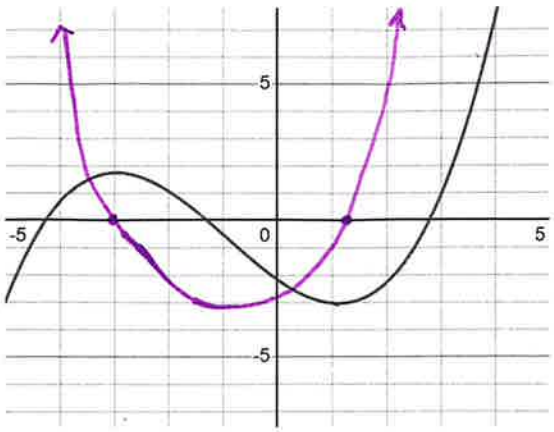
$$\lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} \to \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}$$

$$\lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2} = \frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^4}$$

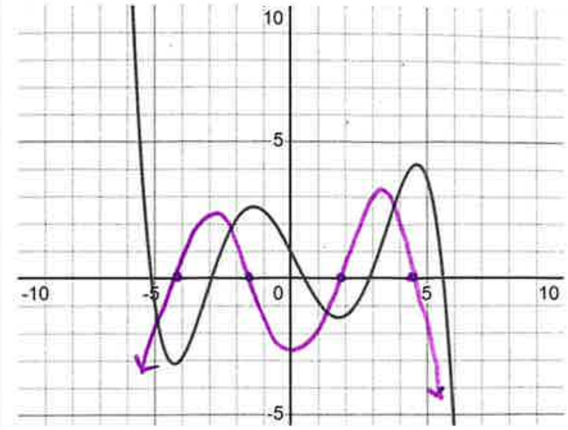
$$\boxed{\frac{-2}{x^3}}$$

13. Use the given graph of f to sketch the graph of f' overtop.

a)



b)



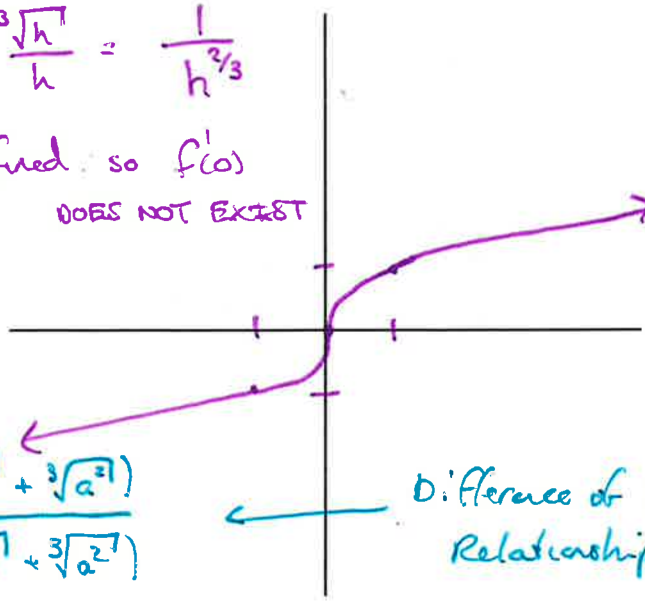
14.

a) Sketch the graph of the cube root function $f(x) = \sqrt[3]{x}$
b) Show that f is not differentiable at 0.
c) If $a \neq 0$, find $f'(a)$.

$h^{1/3}$

b) $f'(0) = \lim_{h\to 0} \frac{\sqrt[3]{0+h} - \sqrt{0}}{h} = \frac{\sqrt[3]{h}}{h} = \frac{1}{h^{2/3}}$

$\frac{1}{h^{2/3}}$ is undefined so $f'(0)$ DOES NOT EXIST



c) $f'(a)$ if $a \neq 0$

$\lim_{h\to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h}$

$\lim_{h\to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h} \cdot \frac{\left(\sqrt[3]{(a+h)^2} + \sqrt[3]{a^2+ah} + \sqrt[3]{a^2}\right)}{\left(\sqrt[3]{(a+h)^2} + \sqrt[3]{a^2+ah} + \sqrt[3]{a^2}\right)}$

Difference of Cube Relationships

$\lim_{h\to 0} \frac{a+h-a}{h(}$

$\lim_{h\to 0} \frac{h}{h(\text{Denominator})} \rightarrow \frac{1}{\sqrt[3]{a^2} + \sqrt[3]{a^2} + \sqrt[3]{a^2}} = \boxed{\frac{1}{3\sqrt{a^2}}}$

15.

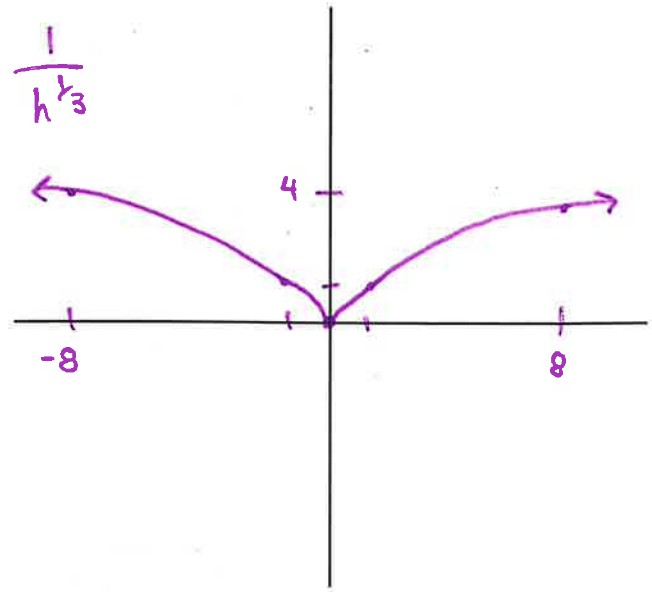
a) Show that the function $f(x) = x^{\frac{2}{3}}$ is not differentiable at 0.

b) Sketch the curve $y = x^{\frac{2}{3}}$

$$f'(0) = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0^{2/3}}{h} = \frac{h^{2/3}}{h} = \frac{1}{h^{1/3}}$$

as $h \to 0$

this DOES NOT EXIST



16. A function is defined by the following conditions:

$f(x) = |x|$ if $-1 \leq x \leq 1$

$f(x+2) = f(x)$ for all values of x

this means the shape repeats every 2 in both directions

a) Sketch the graph of f.

b) For what values of x is f not differentiable?

x is not diffentiable at every integer value ..., −3, −2, −1, 0, 1, 2, 3, ...