1.7 Infinite Series

It may seem to not make sense to talk about adding up infinitely many numbers. It may seem like this is not possible, but there are situations where we implicitly use infinite sums. Consider the fractional equivalent for $0.\overline{4} = \frac{4}{9}$.

$$0.\overline{4} = 0.444\,444\,444\,...$$
$$\frac{4}{9} = \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10\,000} + \cdots$$

Ex.

Aristotle related one of Zeno's paradoxes as: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended."

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

Letting S_n be the sum of the first n terms of the series we have the following

sum of first two terms

sum of first three terms

etc.

$$S_1 = \frac{1}{2} = 0.5$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$$
$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$$
$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.968\,75$$
$$S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984\,375$$
$$S_7 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.992\,187\,5$$
$$\vdots$$
$$S_{10} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} \doteq 0.999\,023\,44$$
$$\vdots$$
$$S_{16} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{16}} \doteq 0.999\,984\,74$$

It seems like as more terms are added, the partial sums become closer and closer to 1. In the limit of an infinite series of terms that the sum is 1.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Or written as a limit

$$\lim_{n\to\infty} S_n = 1$$

In general, for any series t_n the **partial sums** are defined as

$$\left.\begin{aligned} S_1 &= t_1 \\ S_2 &= t_1 + t_2 \\ S_3 &= t_1 + t_2 + t_3 \\ S_4 &= t_1 + t_2 + t_3 + t_4 \end{aligned}\right\} \text{Analyze}$$

$$\vdots$$

n^{th} Partial Sum $\rightarrow$ $S_n = t_1 + t_2 + t_3 + \cdots + t_n$

If the infinite sequence of partial sums has a limit L, then we say the **sum** of the series is L.

$$t_1 + t_2 + t_3 + \cdots + t_n = L$$

or $\lim_{n\to\infty} S_n = L$

Written in sigma notation we have

Greek letter Sigma, means sum $\rightarrow$

$$\sum_{n=1}^{\infty} t_n = L$$

Limit Does Not Exist

A **convergent** series is one that has a sum and a **divergent** series is one that does not have a sum.

↑ Limit exist or sum approaches a number

Ex. 1
Determine whether the following series are convergent or divergent.

(a) $1 + 1 + 1 + 1 + \cdots + 1 + \cdots$
(b) $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$

a) $S_n = \underbrace{1+1+1+\ldots+1}_{n \text{ terms}} = n$

$n \to \infty$

which is not a number

So:

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n$

DNE

$\boxed{\text{DIVERGENT}}$

b) $S_1 = 1$

$S_2 = 1 + (-1) = 0$

$S_3 = 1 + (-1) + 1 = 1$

$S_4 = 1 + (-1) + 1 + (-1) = 0$

Partial sums do not approach a single value

they flip flop between 0 and 1

So, the Sum is DIVERGENT

Ex. 2
Find the sum of the geometric series when it exists
$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \qquad (a \neq 0)$$

The n^{th} partial sum:

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

PC 12: The sum of a finite geometric series with first term a and common ratio $r \neq 1$ is: $S_n = \frac{a(1-r^n)}{1-r}$

Case 1: $|r| < 1$ or $-1 < r < 1$ (Section 1.6)

$\lim_{n\to\infty} r^n = 0$

So, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$= \frac{a(1-0)}{1-r} = \boxed{\frac{a}{1-r}}$

For $|r| < 1$, geometric series is convergent and the sum is
$$S = \frac{a}{1-r}$$

Case 2: $(r = 1)$

$S_n = a + a + a + \ldots$ which is divergent

Case 3: $(r = -1)$

$a - a + a - a + \ldots$ which is divergent

Case 4: $|r| > 1$ or $r > 1$ $r < -1$

$\lim_{n\to\infty} r^n$ DNE so $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$, DNE

Series is Divergent

If $|r| < 1$, the infinite geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \qquad (a \neq 0)$$

Is **convergent** and has the sum

$-1 < r < 1$

$$S = \frac{a}{1-r}$$

In sigma notation the sum can be written

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is **divergent**.

Ex. 3
Find the sum of the series

$$16 - 12 + 9 - \frac{27}{4} + \frac{81}{16} - \cdots$$

First find r : $\frac{-12}{16} = -3/4$ $\quad -1 < r < 1$ is accurate

$$S = \frac{a}{1-r}$$

$$S = \frac{16}{1-(-3/4)} = \frac{16}{1+3/4} = \frac{16}{\frac{4}{4}+3/4} = \frac{16}{7/4} = 16 \cdot \frac{4}{7}$$

$$= \boxed{\frac{64}{7}}$$

Ex. 4

Express the repeating decimal $2.1\overline{35}$ as a fraction

2.1353535

$$2.1\overline{35} \rightarrow 2.1 + \frac{35}{1000} + \frac{35}{100\,000} + \frac{35}{10\,000\,000} + \cdots$$

$$a = \frac{35}{1000}$$

$$r = \frac{\frac{35}{100\,000}}{\frac{35}{1000}} = \frac{\cancel{35}}{100000} \cdot \frac{1000}{\cancel{35}}$$

$$\boxed{\frac{1}{100} = 0.01}$$

$$2.1\overline{35} = 2.1 + \frac{a}{1-r}$$

$$= 2.1 + \frac{\frac{35}{1000}}{1-0.01}$$

$$= \frac{21}{10} + \frac{35}{1000(0.99)}$$

$$= \frac{21}{10} + \frac{35}{990}$$

$$= \frac{2079}{990} + \frac{35}{990} \rightarrow \frac{2114 \div 2}{990 \div 2} = \frac{1057}{495}$$

$$2.1\overline{35} = \boxed{\frac{1057}{495}}$$

Homework Assignment

- Exercise 1.7: #1aceg, 2, 3aceg, 4ac, 5

Chapter Review

- Exercise 1.8: #1aceg, 2, 3aceg, 4aceg, 5, 8, 10 – 14