Exercise 1.7 – Practice Problems

1. Find the sum of each of the following series or state that the series is divergent.

(a) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

$r=\frac{1}{3}$

$S=\frac{1}{1-\frac{1}{3}}=\boxed{\frac{3}{2}}$

(b) $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$

$r=-2/3$

$\frac{1}{1-(-2/3)}=\frac{1}{1\,2/3}=\boxed{\frac{3}{5}}$

(c) $\frac{1}{4}-\frac{5}{16}+\frac{25}{64}-\frac{125}{256}+\cdots$

$r=-5/4$ so Divergent

(d) $\cancel{1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots}$ $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\cdots$

$r=\frac{1}{5}$

$\frac{3}{1-\frac{1}{5}}=\frac{3}{4/5}=\boxed{15/4}$

(e) $1-2+4-8+\cdots$

$r=-2$ so Divergent

(f) $60+40+\frac{80}{3}+\frac{160}{9}+\cdots$

$r=2/3$

$S=\frac{60}{1-2/3}=\frac{60}{1/3}=\boxed{180}$

(g) $0.1+0.05+0.025+0.0125+\cdots$

$r=\frac{1}{2}$

$S=\frac{0.1}{1-1/2}=\frac{0.1}{0.5}=\boxed{0.2}$

(h) $-3+3-3+3-3+\cdots$

$r=-1$ so Divergent

2. Find the sum of each of the following series.

(a)

$$\sum_{n=1}^{\infty}2\left(\frac{3}{4}\right)^{n-1}$$

$a=2$

$r=3/4$

$S=\frac{2}{1-3/4}$

$=2/\tfrac{1}{4}$

$=\boxed{8}$

(b)

$$\sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^{n}$$

$n=1$ $a=-2/5$

$r=-2/5$

$S=\frac{-2/5}{1-(-2/5)}=\frac{-2/5}{7/5}$

$=\boxed{-2/7}$

3. Express the following repeating decimals as fractions.

(a) $0.\overline{1}$

$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots$

$r = \frac{1}{10}$

$S = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \boxed{\frac{1}{9}}$

(b) $0.\overline{25}$

$\frac{25}{100} + \frac{25}{10\,000} + \frac{25}{1\,000\,000} + \dots$

$a = \frac{25}{100}$

$r = \frac{1}{100}$

$S = \frac{\frac{25}{100}}{1 - \frac{1}{100}} = \frac{\frac{25}{100}}{\frac{99}{100}}$

$\boxed{\frac{25}{99}}$

(c) $0.\overline{41}$

$\frac{41}{100} + \frac{41}{10000} + \dots$

$r = \frac{1}{100}$

$a = \frac{41}{100}$

$\frac{\frac{41}{100}}{1 - \frac{1}{100}} = \frac{\frac{41}{100}}{\frac{99}{100}}$

$\boxed{\frac{41}{99}}$

(d) $0.\overline{157}$

$\frac{157}{1000} + \frac{157}{1000000} + \dots$

$a = \frac{157}{1000}$

$r = \frac{1}{1000}$

$\frac{\frac{157}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{157}{1000}}{\frac{999}{1000}}$

$\boxed{\frac{157}{999}}$

(e) $1.1\overline{23}$

$1.1 + \frac{23}{1000} + \frac{23}{100\,000} + \dots$

$a = \frac{23}{1000}$

$r = \frac{1}{100}$

$1.1 + \frac{\frac{23}{1000}}{1 - \frac{1}{100}}$

$= \boxed{\frac{556}{495}}$

(f) $2.3\overline{456}$

Follow the same process

$\boxed{\frac{7811}{3330}}$

d/c ← gives improper

$\boxed{a\ b/c}$ changes decimal to fraction

(g) $0.429\,\overline{113}$

$\boxed{\frac{107\,171}{249\,750}}$

(h) $6.814\,\overline{72}$

$\boxed{\frac{37\,481}{5500}}$

4. For what values of x are the following series convergent? In each case find the sum of the series for those values of x.

(a) $1 + x + x^2 + x^3 + \cdots$

need $\left|\frac{x}{1}\right| < 1$ $|x| < 1$

$r = x$

$S = \boxed{\dfrac{1}{1-x}}$

(b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots$

need $\left|\frac{x}{3}\right| < 1$ $|x| < 3$

$r = \frac{x}{3}$ $S = \dfrac{1}{1-\frac{x}{3}} = \dfrac{1}{\frac{3-x}{3}} = \boxed{\dfrac{3}{3-x}}$

(c) $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

$\left|\frac{1}{x}\right| < 1$ $1 < |x|$

$r = \frac{1}{x}$ $S = \dfrac{1}{1-\frac{1}{x}} \rightarrow \dfrac{1}{\frac{x-1}{x}} \rightarrow \boxed{\dfrac{x}{x-1}}$

(d) $1 + (x-4) + (x-4)^2 + (x-4)^3 + \cdots$

$|x-4| < 1$ $|x| < 5$

$r = (x-4)$

$S = \dfrac{1}{1-(x-4)} = \boxed{\dfrac{1}{5-x}}$

(e) $\sum_{n=1}^{\infty} 2^n x^n$

$n = 1$ $a = 2x$

$n = 2$ $t_2 = 4x^2$

$n = 3$ $t_3 = 8x^3$

$2x + 4x^2 + 8x^3 + \ldots$ $r = 2x$

$|2x| < 1$ $|x| < \frac{1}{2}$

$\boxed{S = \dfrac{2x}{1-2x}}$

5. The series

$$1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \cdots + \frac{(-1)^{n-1}}{n^6} + \cdots$$

is not a geometric series. Use your calculator to find the first eight partial sums of this series. Does it appear that this series is convergent? If so, estimate its sum to five decimal places.

$S_1 = 1$

$S_2 = 0.984375$

$S_3 = 0.985747$

$S_4 = 0.985503$

$S_5 = 0.985567$

$S_6 = 0.985545$

$S_7 = 0.985554$

$S_8 = 0.985550$

Series seems to converge at:

$\boxed{0.98555}$

6. The series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots$$

is not a geometric series.

a) Use your calculator to find its first 15 partial sums.

$S_1 = 0.5$ $S_6 = 0.8571$ $S_{10} = 0.9091$ $S_{14} = 0.9333$

$S_2 = 0.6667$ $S_7 = 0.875$ $S_{11} = 0.9167$ $S_{15} = 0.9375$

$S_3 = 0.75$ $S_8 = 0.8889$ $S_{12} = 0.9231$

$S_4 = 0.8$ $S_9 = 0.9$ $S_{13} = 0.9286$

$S_5 = 0.8333$

b) Use the identity

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

To find an expression for the nth partial sum S_n.

t_1

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

c) Use part b) to find the sum of the series

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1-\frac{1}{\infty} = 1-0 = \boxed{1}$$

d) How many terms of the series would be required so that the partial sum differs from the total sum by less than 0.001?

Total sum $= 1$

$$1-\left(1-\frac{1}{n+1}\right) < 0.001$$

$$1-1+\frac{1}{n+1} < 0.001$$

$$\frac{1}{n+1} < 0.001$$

$$1 < 0.001n + 0.001$$

$$0.999 < 0.001n$$

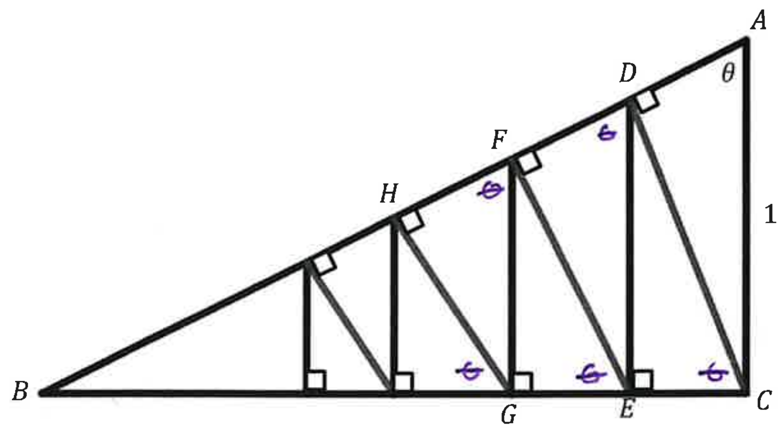
$$999 < n$$

at least 1000 terms needed

7. A right-angle triangle is given with $\angle A = \theta$ and $AC = 1$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF is perpendicular to AB, and this process is continued indefinitely as in the figure below. Find the total length of all the perpendiculars

$$CD + DE + EF + FG + \cdots$$

In terms of θ.



$\sin\theta = \frac{\overline{CD}}{1}$

$\sin\theta = \frac{DE}{CD}$

$CD\sin\theta = DE$

$EF = \sin\theta\, DE$

$\angle A = \theta$

$\overline{CD} = \sin\theta$

$\overline{DE} = CD\sin\theta = \sin\theta\sin\theta = \sin^2\theta$

$\overline{EF} = DE\sin\theta = \sin^2\theta\sin\theta = \sin^3\theta$

$r = \sin\theta$

$a = \sin\theta$

$$\boxed{S = \frac{\sin\theta}{1-\sin\theta}}$$