1.6 Infinite Sequences

A **sequence** is defined as a list of numbers written in a definite order:

$$t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$$

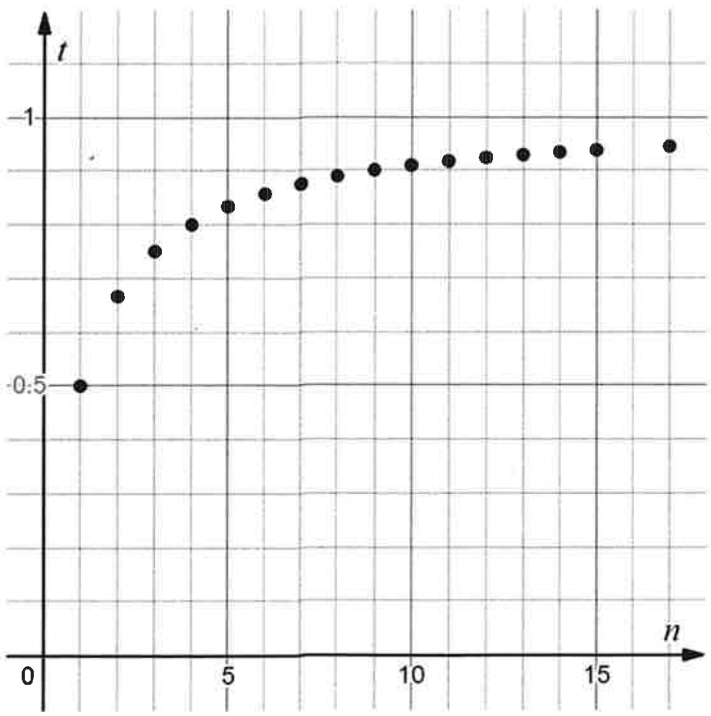
The *first term* is t_1, the *second term* is t_2, and so on to the *nth term*. We will only considering infinite sequences, such that t_n is succeeded by t_{n+1}.

For every positive integer n there is a corresponding number t_n, so you could think of a sequence as a function whose domain is the set of positive integers. Instead of using function notation and writing $t(n)$ we write t_n for the value of the function at the number n.

Ex. 1

List the first five terms of the sequence defined by

$$t_n = \frac{n}{n+1}$$



it approaches y=1

Notice that the terms in the above sequence are all less than 1 because $n < n + 1$. They get closer and closer to 1 as n increases. If we take the limit as n approaches infinity, we the sequence *equals* 1.

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

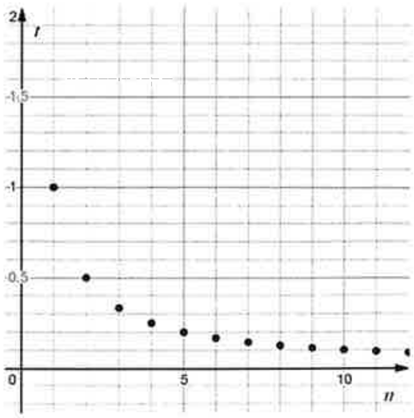
And in general, for any sequence t_n

$$\lim_{n\to\infty} t_n = L$$

Ex. 2

Find $\lim_{n \to \infty} \frac{1}{n}$.

$t_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$



As n increases $\frac{1}{n}$ decreases.

So much so we say

The previous example demonstrates a general fact for sequences:

$$\boxed{\lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0}$$

In addition to this rule, the properties of limits described in Section 1.2 are also valid for limits of sequences.

Ex. 3

Find $\lim_{n \to \infty} \frac{n^2 - n}{2n^2 + 1}$.

DS: $\frac{\infty}{\infty}$

Since we can't factor, we divide out the highest power of n.

Highest power is: n^2

* See Horizontal Asymptote in PC 11 and PC 12 for Rational Functions *

$$\lim_{n \to \infty} \frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{\frac{2n^2}{n^2} + \frac{1}{n^2}} \rightarrow \frac{1 - \frac{1}{n}}{2 + \frac{1}{n}} \rightarrow \lim_{n \to \infty} \frac{1 - \frac{1}{\infty}}{2 + \frac{1}{\infty}} \rightarrow \frac{1 - 0}{2 + 0} = \boxed{\frac{1}{2}}$$

Ex. 4
Find the following limits if they exist.

(a) $\lim_{n\to\infty}(-1)^n$

$t_n = (-1)^n$

$t_n = -1, 1, -1, 1, \ldots$

they approach no constant value.

$\lim_{n\to\infty}(-1)^n = DNE$

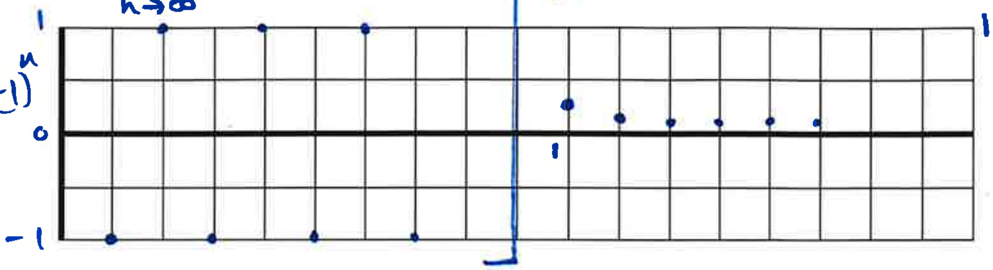
(b) $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n$

$\frac{1}{2^n}$

$t_n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

As n increases $\frac{1}{2^n}$ decreases

$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$



From example 4(b) the following result can be reasoned.

$$\boxed{\text{If } |r| < 1, \text{ then } \lim_{n\to\infty} r^n = 0}$$

r needs to be:

$-1 < r < 1$

in order to converge to 0.

Ex. 5
The **Fibonacci sequence** is defined recursively by the equations.

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \qquad (n \geq 3)$$

Find the first eight terms of the sequence.

$f_1 = 1$

$f_2 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = 5 + 3 = 8$

$f_7 = 8 + 5 = 13$

$f_8 = 13 + 8 = 21$

Homework Assignment

- Exercise 1.6: #1, 2, 3odd, 4, 5, 7, 9