Exercise 1.6 – Practice Problems

1. State the limit of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n$

$\lim_{n \to \infty} \frac{1}{3^n} \to \frac{1}{\infty} = \boxed{0}$

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4+\frac{1}{n}, \ldots$

$\lim_{n \to \infty} 4 + \frac{1}{\infty} \to \boxed{4}$

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$

DOES NOT EXIST

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$

$\boxed{3}$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

Both terms approach 0

so $\boxed{0}$

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6} \ldots, 6+\frac{(-1)^n}{n}, \ldots$

$\lim_{n \to \infty} 6 + \frac{(-1)^{\infty}}{\infty} \to 6+0 = \boxed{6}$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5} \ldots$

odd terms approach 1
even terms approach 0

so $\boxed{\text{DNE}}$

2. List the first six terms of the following sequences

(a) $t_n = \dfrac{n-1}{2n-1}$

$0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$

(b) $t_n = \dfrac{2n}{n^2+1}$

$1, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$

$\frac{6}{10} \to \frac{3}{5}$, $\frac{10}{26} \to \frac{5}{13}$

(c) $t_n = n2^n$

$2, 8, 24, 64, 160, 384$

(d) $t_n = \dfrac{(-1)^{n-1}}{n}$

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$

(e) $t_1 = 1, t_n = \dfrac{1}{1+t_{n-1}}\ (n \geq 2)$

$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{9}, \frac{8}{13}$

(f) $t_1 = 1, t_2 = 2, t_n = t_{n-1} - t_{n-2}\ (n \geq 3)$

$1, 2, 1, -1, -2, -1$

3. Find the following limits or state that the limit does not exist.

(a) $\lim\limits_{n\to\infty} \dfrac{1}{n^2}$

$\frac{1}{\infty} \to \boxed{0}$

(b) $\lim\limits_{n\to\infty} \dfrac{1}{n+5}$

$\frac{1}{\infty} \to \boxed{0}$

(c) $\lim\limits_{n\to\infty} \left(6 + \dfrac{1}{n^3}\right)$

$6 + \frac{1}{\infty} \to \boxed{6}$

(d) $\lim\limits_{n\to\infty} \dfrac{n}{3n-1}$

$\dfrac{\frac{n}{n}}{\frac{3n}{n} - \frac{1}{n}} \to \dfrac{1}{3 - \frac{1}{\infty}} = \boxed{\frac{1}{3}}$

(e) $\lim\limits_{n\to\infty} \dfrac{6n+9}{3n-2}$

$\dfrac{\frac{6n}{n}+\frac{9}{n}}{\frac{3n}{n}-\frac{2}{n}}$

$\dfrac{6+\frac{9}{\infty}}{3-\frac{2}{\infty}} \to \dfrac{6+0}{3-0} = \dfrac{6}{3} = \boxed{2}$

(f) $\lim\limits_{n\to\infty} 5n$

$\boxed{\text{DNE}}$

(g) $\lim\limits_{n\to\infty} \dfrac{n^2+1}{2n^2-1}$

$\dfrac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{2n^2}{n^2}-\frac{1}{n^2}}$

$\dfrac{1+\frac{1}{\infty}}{2-\frac{1}{\infty}} \to \dfrac{1+0}{2-0} = \boxed{\frac{1}{2}}$

(h) $\lim\limits_{n\to\infty} \dfrac{(n+1)^2}{n(n+2)}$

$\dfrac{n^2+2n+1}{n^2+2n}$

$\dfrac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}} \to \dfrac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{2}{n}} \to \dfrac{1+0+0}{1+0}$

$\boxed{1}$

(i) $\lim\limits_{n\to\infty} \dfrac{(-1)^{n+1}}{n}$

sequence: $1, -\frac{1}{2}, \frac{1}{3}, \ldots$

$\lim\limits_{n\to\infty} = \boxed{0}$

(j) $\lim\limits_{n\to\infty} \left(-\frac{1}{4}\right)^n$

as $4^n \to \infty$ as $n \to \infty$

$-\dfrac{1}{4^n} \to \boxed{0}$

(k) $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1}$

$\dfrac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} \to \dfrac{\frac{1}{n}}{1+\frac{1}{n^2}} \to \dfrac{0}{1} = \boxed{0}$

(l) $\lim\limits_{n\to\infty} (-1)^{n-1}\cdot n$

$1, -2, 3, \ldots$

$\lim\limits_{n\to\infty}$ DNE

(m) $\lim\limits_{n\to\infty} 5^{-n}$

$\lim\limits_{n\to\infty} \dfrac{1}{5^n} \to \boxed{0}$

(n) $\lim\limits_{n\to\infty} (n^3+n^2)$

$\boxed{\text{DNE}}$

(o) $\lim\limits_{n\to\infty} \dfrac{1+n-2n^2}{1-n+n^2}$

$\dfrac{\frac{1}{n^2}+\frac{n}{n^2}-\frac{2n^2}{n^2}}{\frac{1}{n^2}-\frac{n}{n^2}+\frac{n^2}{n^2}}$

$\dfrac{0+0-2}{0-0+1} = \boxed{-2}$

(p) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}}$

$\dfrac{1}{\sqrt{\infty}} \to \boxed{0}$

(q) $\lim_{n\to\infty}\frac{1}{n^5}$

$\frac{1}{\infty} \to \boxed{0}$

(r) $\lim_{n\to\infty}\frac{1-n^3}{1+2n^3}$

$\dfrac{\frac{1}{n^3}-\frac{n^3}{n^3}}{\frac{1}{n^3}+\frac{2n^3}{n^3}} \to \frac{0-1}{0+2} = \boxed{-\frac{1}{2}}$

(s) $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n$

$\frac{2^n}{3^n}$ this gets bigger infinitely faster

$\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = \boxed{0}$

(t) $\lim_{n\to\infty}\left(\frac{4}{3}\right)^n$

opposite scenario so

$\boxed{\text{DNE}}$

4. If $t_1 = 0.3, t_2 = 0.33, t_3 = 0.333, t_4 = 0.3333$ and so on, what is: $\lim_{n\to\infty} t_n$

$t_n = 0.3333333\ldots = \boxed{\frac{1}{3}}$

5. If

$$t_n = \frac{2^n}{n^2}$$

Use your calculator to find t_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50, \textit{and } 100$

Does the limit below exist? If so, guess its value:

$$\lim_{n\to\infty}\frac{2^n}{n^2}$$

$t_1 = 2$

$t_2 = 1$

$t_3 = \frac{8}{9}$

$t_4 = 1$

$t_5 = \frac{32}{25}$

$t_6 = \frac{16}{9}$

$t_7 = \frac{128}{49}$

$t_8 = 4$

$t_9 = \frac{512}{81}$

$t_{10} = \frac{1024}{100}$

$t_{20} \doteq 2621.44$

$t_{50} = 4.5\times10^{11}$

$t_{100} = 1.28\times10^{26}$

constantly increasingly. so Limit DNE

6. If

$$t_n = \sqrt[n]{n}$$

Use your calculator to find t_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50, 100, 500, 1000$ *and* $10\,000$

Then guess the value of the limit.

$t_1 = 1$
$t_2 = \sqrt{2} = 1.414214$
$t_3 = \sqrt[3]{3} = 1.442250$
$t_4 = \sqrt[4]{4} = 1.41421$
$t_5 = \sqrt[5]{5} = 1.37973$
$t_6 = \sqrt[6]{6} = 1.34801$

$t_7 = \sqrt[7]{7} = 1.3205$
$t_8 = \sqrt[8]{8} = 1.29684$
$t_9 = \sqrt[9]{9} = 1.276518$
$t_{10} = \sqrt[10]{10} = 1.258925$
$t_{20} = \sqrt[20]{20} = 1.16159$
$t_{50} = \sqrt[50]{50} = 1.08138$

$t_{100} = \sqrt[100]{100} = 1.047129$
$t_{500} = \sqrt[500]{500} = 1.012507$
$t_{1000} = 1.0069317$
$t_{10000} = 1.000921$

$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$

7. Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will there be in the nth month. Show that the answer is f_n, the nth term of the Fibonacci sequence defined in Example 5 in your notes.

let f_n be the number of rabbit pairs in the n^{th} month.

We know

$f_1 = 1$ and they reproduce in the second month so,

$f_2 = 1$

In the n^{th} month every pair that is at least 2 months old (existed in f_{n-2}) will add a pair to the f_{n-1} that already exist.

$f_n = f_{n-1} + f_{n-2}$ ← this is the Fibonacci Sequence

8. Find the limit of the following sequence by expressing each term as a power of 2

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

$2^{\frac{1}{2}}, (2\cdot 2^{\frac{1}{2}})^{\frac{1}{2}}, (2(2(2^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$

$2^{\frac{1}{2}}, 2^{\frac{1}{2}}\cdot 2^{\frac{1}{4}}, 2^{\frac{1}{2}}(2^{\frac{1}{4}})(2^{\frac{1}{8}})$

$2^{\frac{1}{2}}, 2^{3/4}, 2^{\frac{7}{8}}, 2^{\frac{15}{16}}$

You get $2^{\frac{2n-1}{2n}}$

$2^{\frac{2n}{2n}}\cdot 2^{-\frac{1}{2n}}$

$2(2^{-\frac{1}{\infty}}) \to 2(2^0) = \boxed{2}$

9.

(a) A sequence is defined recursively by:

$$t_1 = 1, \quad t_n = \frac{1}{2t_{n-1}+1} \qquad (n \geq 2)$$

Find t_2, t_3, t_4, t_5, t_6 and then guess the value of $\lim_{n\to\infty} t_n$

$t_1 = 1$

$t_2 = \frac{1}{3}$

$t_3 = 3/5$

$t_4 = \frac{5}{11}$

$t_5 = \frac{11}{21}$

$t_6 = \frac{21}{49}$

all almost half

$\lim_{n\to\infty} t_n = \frac{1}{2}$

(b) Assume that $\lim_{n\to\infty} t_n = L$ exists. What is the value of $\lim_{n\to\infty} t_{n-1}$?
Find the value of L by taking the limit of both sides of the recursive equation.

If $t_n = L$ as $\lim_{n\to\infty}$

then $t_{n-1} = L$

$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{2t_{n-1}+1} = L$

$t_n = \lim_{n\to\infty} \frac{1}{2L+1} = L$

$1 = L(2L+1)$

$1 = 2L^2 + L$

$0 = 2L^2 + L - 1$

$0 = (2L-1)(L+1)$

Limit has to be positive

$L = \frac{1}{2}$ or

$L = -1$