1.5 Velocity and Other Rates of Change

- **Rate of Change** is defined as how quickly one variable changes with respect to time.
- **Average Speed** is defined as the distance (d) travelled divided by the time elapsed. In other words, the rate at which distance changes with time.
- **Instantaneous Speed** is defined as the speed with which an object is moving at an instant in time.
- **Average Velocity ($\overline{v}$)** is defined as the displacement (distance and direction, Δs) divided by the time elapsed. Note that if there is no change in direction both the average velocity and average speed will be the same.
- **Instantaneous Velocity (v)** is defined as the velocity (speed and direction) with which an object is moving at an instant in time.

Latin s for spatium (spacial location)

Tangent Line Slope

Ex.
Calculate the average speed and average velocity of a car that is driven on the highway north for three hours and covers a distance 270 km.

Since direction stays the same Ave Speed = Ave Velocity

$$\text{Ave Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{270\text{ km}}{3\text{ hr}} = \boxed{90\text{ km/h}}$$

Ex. 1
Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. How fast is the ball falling after 3 s? Assume the ball is dropped from rest (initial speed is zero). The distance the ball has fallen is measured from the top of the tower and the downward direction is taken to be negative.

Instantaneous velocity

$$s(t) = \cancel{v_0 t}^{\,0} + \frac{1}{2}at^2$$

initial velocity is 0

$$s(t) = \frac{1}{2}at^2$$

$v_0 = 0\text{ m/s}$

$a = -9.8\text{ m/s}^2$ ← gravity

$$s(t) = \frac{1}{2}(-9.80)t^2$$

$$= -4.9t^2$$

Estimate velocity over a small range of time.

3s → 3.1 s

$$\overline{v} = \frac{\Delta s}{\Delta t} = \frac{s(3.1) - s(3)}{3.1 - 3} = \frac{-4.9(3.1)^2 - (-4.9)(3)^2}{0.1}$$

$$\Rightarrow \boxed{\overline{v} = -29.89\text{ m/s}}$$

The following table shows the results of successive calculations of the average speed over smaller time periods.

Time Interval	Average Velocity (m/s)
$3 \leq t \leq 4$	-34.3
$3 \leq t \leq 3.1$	-29.89
$3 \leq t \leq 3.05$	-29.645
$3 \leq t \leq 3.01$	-29.449
$3 \leq t \leq 3.001$	-29.4049

The average velocity approaches the value $-29.4\ \mathrm{m/s}$. Consider computing the average velocity over the general time interval $3 \leq t \leq 3 + h$

$$\begin{aligned}
\text{Average Velocity} &= \frac{\Delta s}{\Delta t} \\
&= \frac{s(3+h) - s(3)}{h} \\
&= \frac{-4.9(3+h)^2 - [-4.9(3)^2]}{h} \\
&= \frac{-4.9(\cancel{9} + 6h + h^2 - \cancel{9})}{h} \\
&= \frac{-4.9(6\cancel{h} + h^{\cancel{2}})}{h} \\
&\boxed{= -29.4 - 4.9h \qquad \text{if } h \neq 0}
\end{aligned}$$

Back to Limits

Here we are assuming that h is very small and therefore $4.9h$ is very close to zero such that the average velocity at $t = 3\ \mathrm{s}$ is very close to $-29.4\ \mathrm{m/s}$. The instantaneous velocity at 3 s can be calculated by taking the limit as h approaches 0.

$$\begin{aligned}
v &= \lim_{h \to 0}(-29.4 - 4.9h) \\
&\boxed{= -29.4\ \mathrm{m/s}}
\end{aligned}$$

Instaneous.

NOTE

It was not possible to directly substitute $h = 0$ into the original expression for the average velocity as this would result in a meaningless $\frac{0}{0}$ situation. Rather the limit of the average velocity as h approaches zero has been computed.

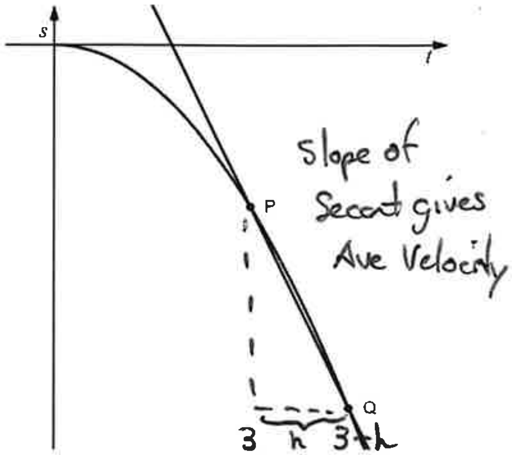
This process of calculating average and instantaneous velocities is closely related to finding the slope of the tangent line. Consider the graph of the position function $s(t) = -4.9t^2$. The slope of the secant line connecting points $P(3, -4.9(3)^2)$ and $Q(3+h, -4.9(3+h)^2)$ on the graph on the next page. The slope of the secant line PQ can be calculated as

$$m_{PQ} = \frac{-4.9(3+h)^2 - [-4.9(3)^2]}{h}$$

Which is the same as the average velocity over the time interval $3 \leq t \leq 3 + h$. If we take the limit as h approaches zero, then the slope calculation above becomes the actual slope of the tangent line at P which also is the instantaneous velocity of the object at P.

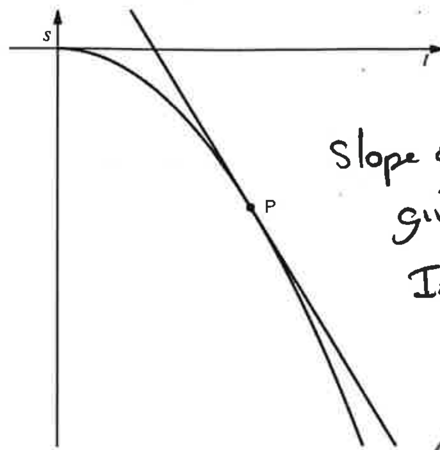
Average Velocity

$s(t) = -4.9t^2$



Instantaneous Velocity

$s(t) = -4.9t^2$



Slope of Tangent gives Instantaneous Velocity

1-4, 5a

In general, if an object moves along a straight-line path according to a **position-function** $s(t)$, its displacement (distance and direction) or **change in position** over the time interval $t = a$ to $t = a + h$ can be calculated as

$$\Delta s = s(a+h) - s(a)$$

And therefore, the average velocity over this time can be calculated using

$$\frac{\Delta s}{\Delta t} = \frac{s(a+h) - s(a)}{h}$$

Which is the slope of the secant line PQ connecting the points $P(a, s(a))$ and $Q(a+h, s(a+h))$. If we take the limit as h approaches zero, the average velocity becomes the instantaneous velocity at P, which is also the slope of the tangent line at point P.

Instantaneous Velocity

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Ex. 2

The position, in metres, of a particle moving in a straight line is given by the function $s(t) = t^2 + 2t$, where t is measured in seconds. Find the velocity of the particle after 3 s.

$v(t)$ is the velocity function

$$v(3) = \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 + 2(3+h) - [3^2 + 2(3)]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{9} + 6h + h^2 + \cancel{6} + 2h - \cancel{15}}{h}$$

$$\lim_{h \to 0} \frac{h^2 + 8h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(h+8)}{\cancel{h}} = 0 + 8 = \boxed{8\,m/s}$$

Other Rates of Change

Suppose y is a function of x such that $y = f(x)$. If x changes from x_1 to x_2, the change in x is

$$\Delta x = \underset{\text{final}}{x_2} - \underset{\text{initial}}{x_1}$$

And the corresponding change in y is

$$\Delta y = \overset{\text{final}}{f(x_2)} - \overset{\text{initial}}{f(x_1)}$$

And the **average rate of change of y with respect to x** over the interval $x_1 \leq x \leq x_2$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The **instantaneous rate of change of y with respect to x** at $x = x_1$ can be calculated as the slope of the tangent to $f(x)$ at the point $x = x_1$. This can be done by taking the limit as Δx approaches zero which is equivalent to letting x_2 approach x_1.

$$\boxed{\text{Rate of Change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

Ex. 3

A thermometer is taken from a room where the temperature is 20°C to the outdoors where the temperature is 5°C. Temperature readings (T) are taken every half-minute and are shown in the following table. The time (t) is measured in minutes. Think units °C/min

t (min)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
T (°C)	20	15	12	9.8	8.3	7.2	6.5	6.0	5.7	5.5	5.3

(a) Find the average of change of temperature with respect to time over the following time intervals:
   (i) $2 \leq t \leq 4$ (ii) $2 \leq t \leq 3.5$
   (iii) $2 \leq t \leq 3.0$ (iv) $2 \leq t \leq 2.5$

(b) Sketch the graph of T as a function of t and use it to estimate the instantaneous rate of change of temperature with respect to time when $t = 2$ min.

a) i) $\frac{\Delta T}{\Delta t} = \frac{T(4) - T(2)}{4-2}$

$= \frac{5.7 - 8.3}{2}$

$= \frac{-2.6}{2}$

$= -1.3\,°C/\text{min}$

ii) $\frac{\Delta T}{\Delta t} = \frac{T(3.5) - T(2)}{3.5 - 2}$

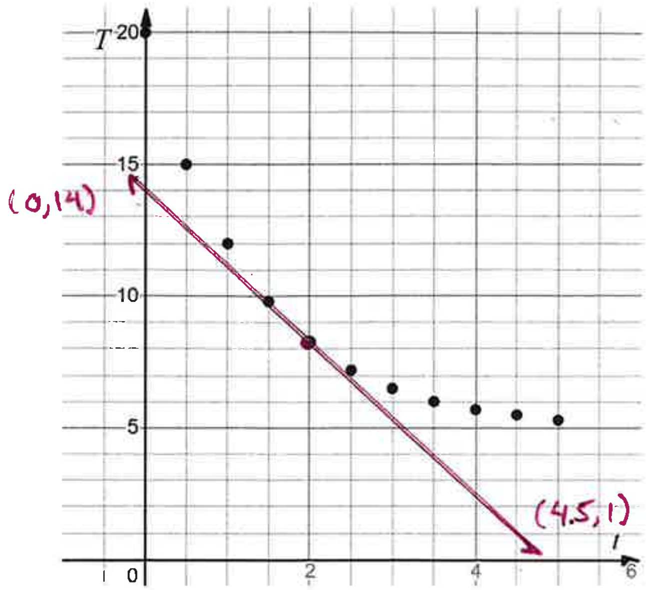
$= \frac{6 - 8.3}{1.5}$

$= \frac{-2.3}{1.5}$

$= -1.5\,°C/\text{min}$

iii) $\frac{\Delta T}{\Delta t} = -1.8\,°C/\text{min}$

iv) $\frac{\Delta T}{\Delta t} = -2.2\,°C/\text{min}$



Pick any two points on the tangent line.

$$\frac{14-1}{0-4.5} = \frac{13}{-4.5} = -2.89\ {}^{\circ}C/\text{min}$$

Remember this is still an approximation.

Ex. 4
A spherical balloon is being inflated. Find the $\boxed{\text{rate of change of the volume}}$ with respect to the radius when the radius is 10 cm.

instantaneous rate

V of Sphere is: $\frac{4}{3}\pi r^3$ $V(r) = \frac{4}{3}\pi r^3$

$$\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \lim_{r \to 10} \frac{V(r) - V(10)}{r-10}$$

$$= \lim_{r \to 10} \frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi (10)^3}{r-10}$$

Factor out constant term

$$\frac{4}{3}\pi \lim_{r \to 10} \frac{r^3 - 10^3}{r-10}$$

$$\frac{4}{3}\pi \lim_{r \to 10} \frac{\cancel{(r-10)}(r^2 + 10r + 10^2)}{\cancel{(r-10)}}$$

$$\frac{4}{3}\pi (10^2 + 10(10) + 100)$$

$$\frac{4}{3}\pi (300) = 400\pi\ \text{cm}^3/\text{cm}$$

$$\doteq 1256.637$$

$$\doteq \boxed{1257\ \text{cm}^3/\text{cm}}$$

DO NOT cancel units

this is a rate of change $\frac{\text{Volume}}{\text{Radius}}$

Homework Assignment

- Exercise 1.5: #1 – 4, 5a, 6, 7, 9