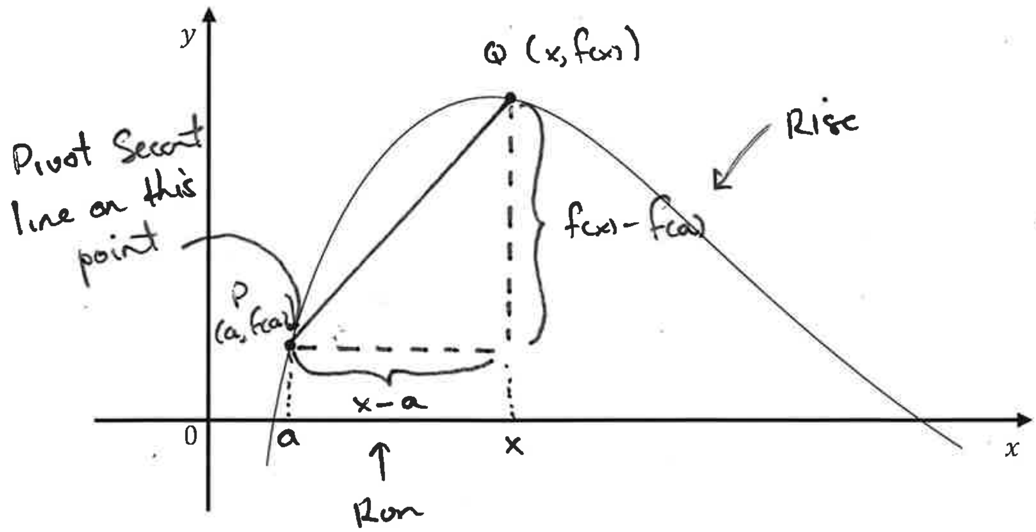
## 1.4 Using Limits to Find Tangents

Consider the curve $C$ defined by the equation $y = f(x)$, if you want to find the slope of the line tangent to $C$ at the point $P(a, f(a))$, then you must first consider the nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line $PQ$.

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

Secant Line: A line connecting two points on a curve.




Tangent Line: Line touching a curve in only on place.

Let $Q$ approach $P$ along the curve $C$ by letting $x$ approach $a$. If $m_{PQ}$ approaches a number $m$, then the tangent is defined to be the line through $P$ with slope $m$. Written using limits

$$\boxed{m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

$\Delta x = x - a$

**Ex. 1**

(a) Find the slope and the equation of the tangent line to the curve $y = 2x^2 + 4x - 1$ at the point $(2, 15)$ ← $a = 2$

(b) Sketch the curve and the tangent line.

a) $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$m = \lim_{x \to a} \frac{(2x^2 + 4x - 1) - [2(2)^2 + 4(2) - 1]}{x - 2}$

$m = \lim_{x \to a} \frac{2x^2 + 4x \cancel{- 1} - 8 - 8 \cancel{+ 1}}{x - 2}$ constants will always cancel

$m = \lim_{x \to a} \frac{2x^2 + 4x - 16}{(x - 2)}$

$m = \lim_{x \to 2} \frac{2(x^2 + 2x - 8)}{x - 2}$

$m = 2 \lim_{x \to 2} \frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}}$

$m = 2 \lim_{x \to 2} (x + 4)$

$m = 2(6) = 12$

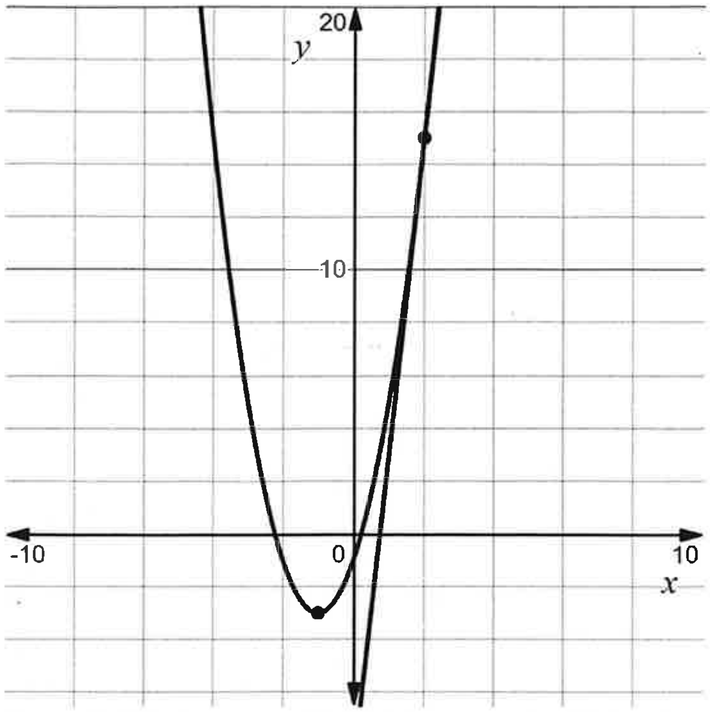
$y = 12x + b$

$15 = 12(2) + b$

$15 = 24 + b$

$b = -9$

$\boxed{y = 12x - 9}$




$y = 2x^2 + 4x - 1$

Roots: Need Quadratic Equation

Complete the Square to find Vertex:

$y = 2x^2 + 4x - 1$

$y = 2(x^2 + 2x) - 1$

$y = 2(x^2 + 2x + 1 - 1) - 1$

$y = 2(x^2 + 2x + 1) - 2 - 1$

$y = 2(x+1)^2 - 3$ $\boxed{\text{Vertex: } (-1, -3)}$

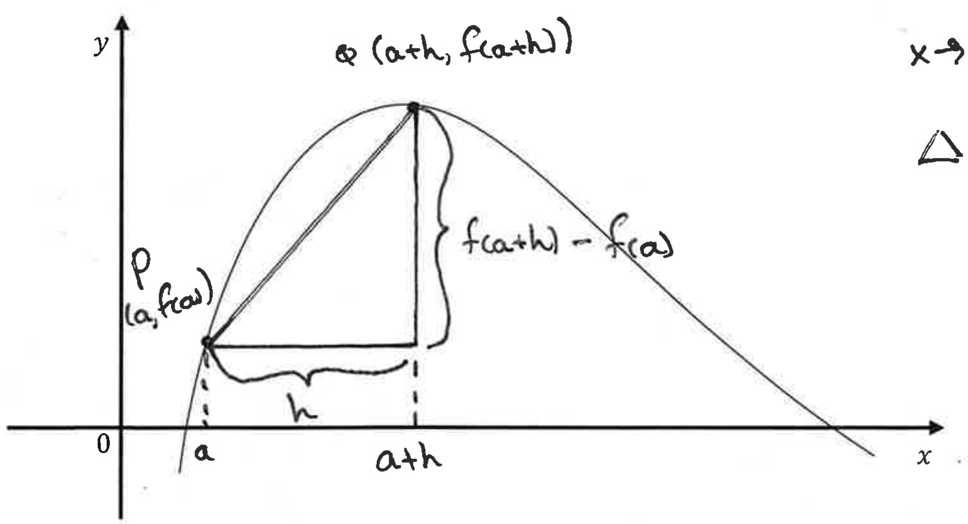
Reconsidering the same curve, another expression for the slope of the secant line $PQ$ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h} \quad *$$

$h = x - a$

$x \to a \Rightarrow \Delta x \to 0$

$\Delta x \to 0$ when $h \to 0$




The slope of the tangent line can be calculated using limits and the above expression as

$$\boxed{m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} \quad *$$

**Ex. 2**

Find the tangent line to the hyperbola $xy = 1$ at the point $\left(-2, -\frac{1}{2}\right)$.

$a = -2$

$xy = 1 \rightarrow \boxed{y = \frac{1}{x}}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h}$$

$$m = \lim_{h \to 0} \frac{\frac{1(-2) - 1(-2+h)}{(-2+h)(-2)}}{h}$$

$$m = \lim_{h \to 0} \frac{\frac{-2+2-h}{-2(-2+h)}}{h}$$

$$m = \lim_{h \to 0} \frac{-h}{-2(-2+h)} \cdot \frac{1}{h}$$

$$m = \lim_{h \to 0} \frac{1}{2(-2+h)}$$

$$m = \frac{1}{2(-2)} = \boxed{-\frac{1}{4}}$$

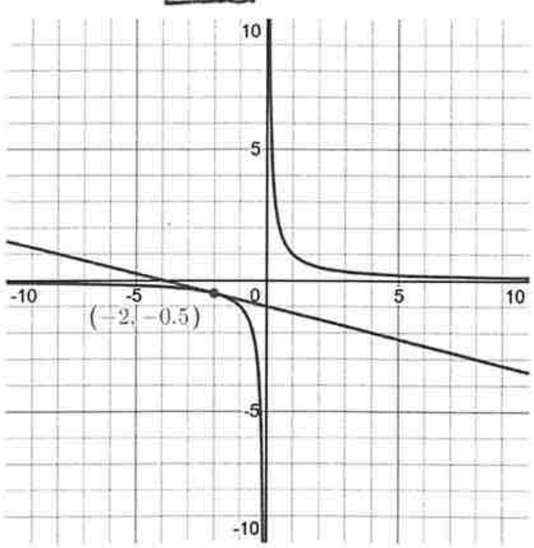
$$y = -\frac{1}{4}x + b$$

$$-\frac{1}{2} = -\frac{1}{4}(-2) + b$$

$$-\frac{1}{2} = \frac{1}{2} + b$$

$$-1 = b$$

$$\boxed{y = -\frac{1}{4}x - 1}$$



Domain restriction $x \geq 2$

**Ex. 3**

(a) Find the tangent line to the curve $y = \sqrt{x-2}$ at the point $(6, 2)$.

(b) Graph the curve and the tangent line.

$f(a+h) = \sqrt{a+h-2}$ $\quad f(a) = \sqrt{a-2}$ $\quad a = 6$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \rightarrow \lim_{h \to 0} \frac{\sqrt{a+h-2} - \sqrt{a-2}}{h} \rightarrow \lim_{h \to 0} \frac{\sqrt{6+h-2} - \sqrt{6-2}}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$ Radicalize $$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$\lim_{h \to 0} \frac{\cancel{4}+h-\cancel{4}}{h(\sqrt{4+h}+2)} \rightarrow \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} \rightarrow \lim_{h \to 0} \frac{1}{\sqrt{4+0}+2} = \boxed{\frac{1}{4}}$$

$y = \frac{1}{4}x + b$ at $(6,2)$

$2 = \frac{1}{4}(6) + b$

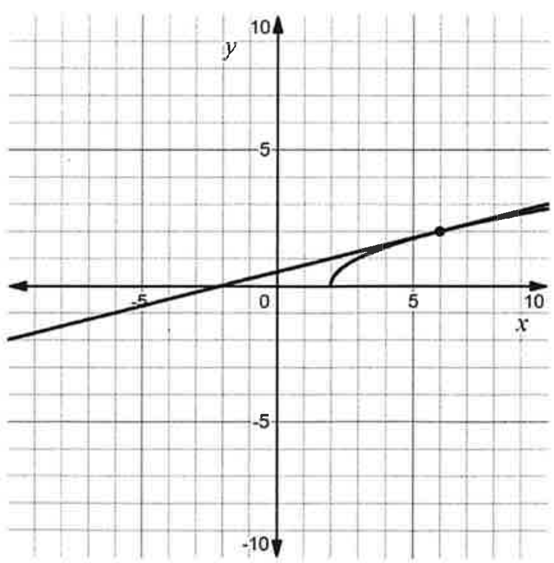
$2 = \frac{3}{2} + b$

$\downarrow$

$\frac{4}{2} - \frac{3}{2} = b$

$b = \frac{1}{2}$

$$\boxed{y = \frac{1}{4}x + \frac{1}{2}}$$




**Homework Assignment**

- Exercise 1.4: #1 – 3, 6, 7 (choose two to graph), 8, 9