## 1.3 One-Sided Limits

Consider the function described below.

we call this a piecewise function

solve it in pieces!

$$f(x) = \begin{cases} x^2, & \text{if } x \le 1 \\ 3 - x, & \text{if } x > 1 \end{cases}$$

Compute $f(0)$, $f(1)$, and $f(2)$

| $f(0)$ | $f(1)$ | $f(2)$ |
|---|---|---|
| $0 \le 1$<br>$f(x) = x^2$<br>$f(0) = 0^2$ $\boxed{f(0) = 0}$ | $1 \le 1$<br>$f(x) = x^2$<br>$f(1) = 1^2$ $\boxed{f(1) = 1}$ | $2 > 1$<br>$f(x) = 3 - x$<br>$f(2) = 3 - 2$<br>$f(2) = 1$ $\boxed{f(2) = 1}$ |

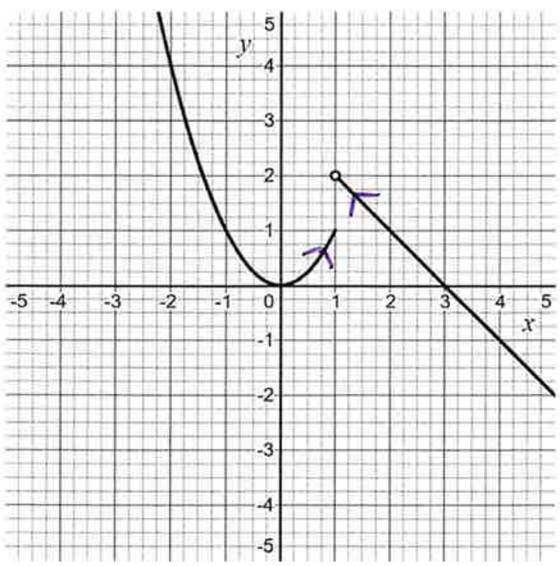
Now evaluate what happens as $x$ approaches 1…

| Approaching from the Left | | Approaching from the Right | |
|---|---|---|---|
| $x < 1$ | $f(x) = x^2$ | $x > 1$ | $f(x) = 3 - x$ |
| 0.9 | 0.81 | 1.1 | 1.9 |
| 0.99 | 0.980 1 | 1.01 | 1.99 |
| 0.999 | 0.998 001 | 1.001 | 1.999 |

Written mathematically, we can say

$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2$$

A better understanding of what is happening with this function as $x$ approaches 1 can be found by looking at the graph of $f(x)$.




$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+}$$

so $\lim_{x \to 1} f(x)$ DNE

Limits for any function $f(x)$ as $x$ approaches some value $a$ can be considered from the left ($x < a$) and from the right ($x > a$). Written mathematically

$$\lim_{x \to a^-} f(x) = L$$

And is read "the limit of $f(x)$ as $x$ approaches $a$ from the **left** equals $L$." Similarly, "the limit of $f(x)$ as $x$ approaches $a$ from the **right** equals $L$" can be written

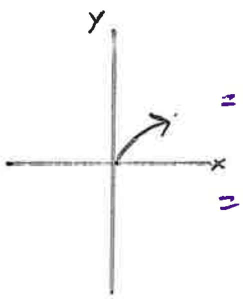
$$\lim_{x\to a^+} f(x) = L$$

Evaluating the limit from both the left and right sides of the number $a$ provides a means of testing whether the limit as $f(x)$ as $x$ approaches $a$ exists.

> If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.
>
> If $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x) = L$.

For computing one-sided limits, the properties presented in the previous section still apply.

**Ex. 1**
Find $\lim_{x\to 0^+} \sqrt{x}$. ← Domain Restriction $\{x \mid x \in \mathbb{R}, x \geq 0\}$

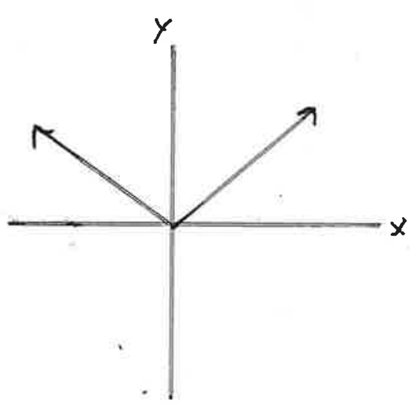



$\lim_{x\to 0^+} \sqrt{x} = \sqrt{\lim_{x\to 0^+} x}$

$= \sqrt{0}$

$= \boxed{0}$

**Ex. 2**
Show that $\lim_{x\to 0}|x| = 0$.




$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{is } x < 0 \end{cases}$

$\lim_{x\to 0^+} |x|$ → $\lim_{x\to 0^+} x$ $= 0$

$\lim_{x\to 0^-} |x|$ $\lim_{x\to 0^-} (-x)$ $= 0$

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x|$

Therefore;

$\lim_{x\to 0} |x| = 0$

Recall DNE means
Does not exist

**Ex. 3**

The **Heaviside function** $H$ is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

This function is named after the electrical engineer Oliver Heaviside (1850 – 1925) and can be used to describe an electric current that is switched on at time $t = 0$. Evaluate, if possible, each of the following.

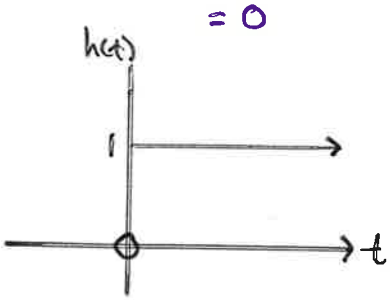
(a) $\lim\limits_{t \to 0^-} H(t)$ (b) $\lim\limits_{t \to 0^+} H(t)$ (c) $\lim\limits_{t \to 0} H(t)$

(a) $= 0$

(b) $= 1$

(c) DNE

limits LHS ≠ lim RHS




**Ex. 4**

Determine whether the limits below exist for the following function.

$$f(x) = \begin{cases} -x - 2, & \text{if } x \leq -1 \\ x, & \text{if } -1 < x < 1 \\ x^2 - 2x, & \text{if } x \geq 1 \end{cases}$$

(a) $\lim\limits_{x \to -1} f(x)$ (b) $\lim\limits_{x \to 1} f(x)$

(a)

$\lim\limits_{x \to -1^-}$

$= \lim\limits_{x \to -1^-} (-x-2)$

$= -(-1)-2$

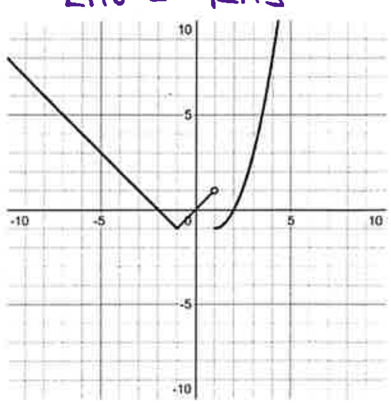
$= \boxed{-1}$

$\lim\limits_{x \to -1^+}$

$= \lim\limits_{x \to -1^+} (x)$

$\boxed{= -1}$

LHS = RHS




$\lim\limits_{x \to -1} f(x) = -1$

(b)

$\lim\limits_{x \to 1^-}$

$= \lim\limits_{x \to 1^-} (x)$

$= \boxed{1}$

$\lim\limits_{x \to 1^+}$

$= \lim\limits_{x \to 1} x^2 - 2x$

$= 1 - 2$

$= \boxed{-1}$

LHS limit ≠ RHS limit

so

$\lim\limits_{x \to 1} f(x)$ DOES NOT EXIST

## Discontinuities

Recall the definition of a continuous function requires that the limit of the function as $x$ approaches $a$ must be equal to the value of the function at that point.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Implicitly this definition requires three things if $f$ is continuous at $a$

1. $f(a)$ is defined (so $a$ is in the domain of $f$)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If $f$ is not continuous at $a$, we say $f$ is **discontinuous** at $a$, or $f$ has a **discontinuity** at $a$.

**Ex. 5**

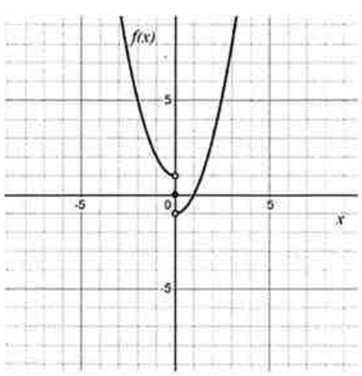
Find where the following functions are discontinuous.

(a) $f(x) = \begin{cases} x^2 + 1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x^2 - 1, & \text{if } x > 0 \end{cases}$ (b) $g(x) = \begin{cases} x + 1, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

$\lim_{x \to 0^-} f(x)$ $= 0^2 + 1$ $= 1$ $\boxed{\neq}$ $\lim_{x \to 0^+} f(x)$ $= 0^2 - 1$ $= -1$

$\lim_{x \to 0} f(x)$ DNE

Discontinuity at $x = 0$

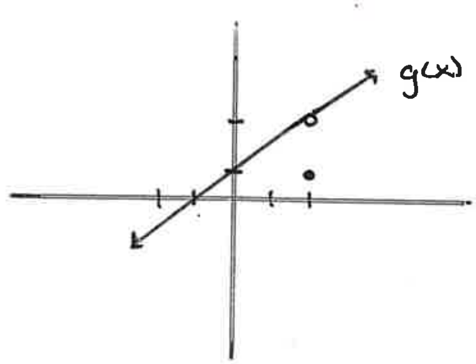



$\lim_{x \to 2} g(x)$ $= (2) + 1$ $= 3$

but... $g(2) = 1$

$\lim_{x \to 2} g(x) \neq g(2)$

Discontinuity at $x = 2$



**Ex. 6**

The cost of a long-distance phone call from Pine Bay to Hester is 26¢ for the first minute and 22¢ for each additional minute (or part of a minute). There is a minimum charge of 34¢ on all calls. Draw the graph of the cost $C$ (in dollars) of a phone call as a function of the time $t$ (in minutes). Where are the discontinuities of this function?

$c(t) = 0.34$ if $0 < t \le 1$

$c(t) = 0.26 + 0.22 = 0.48$ if $1 < t \le 2$

$c(t) = 0.26 + 2(0.22) = 0.70$ if $2 < t \le 3$

$c(t) = 0.26 + 3(0.22) = 0.92$ if $3 < t \le 4$

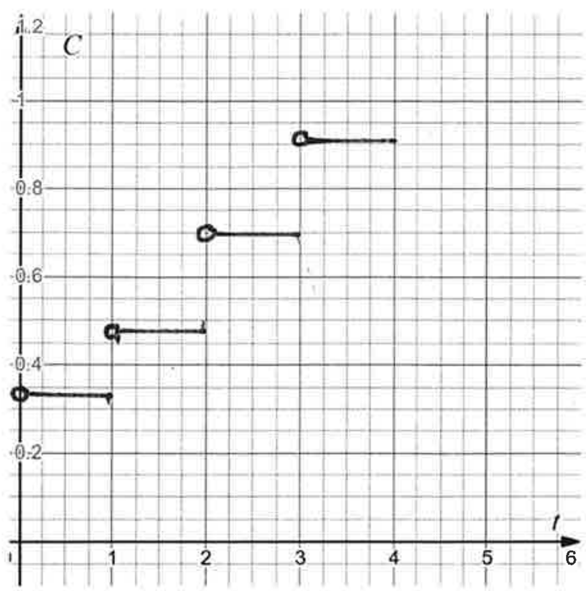
Consider:

$\lim_{t \to 1^-} c(t) = 0.34$

$\lim_{t \to 1^+} c(t) = 0.48$

this pattern continues so we have discontinuities at $t = 1, 2, 3, \ldots$

This is called a STEP FUNCTION




**Homework Assignment**

Exercise 1.3: #1, 3, 4acegi, 5, 7, 8, 10 ac, 11, 14