1.2 The Limit of a Function

From the previous section we saw how limits play a role in calculating the slope of the tangent line to a curve. We will see as we continue through this chapter that limits come into play when calculating velocities and other rates of change. Limits are central to basic calculus, so it is important that we have various methods available to us to compute them.

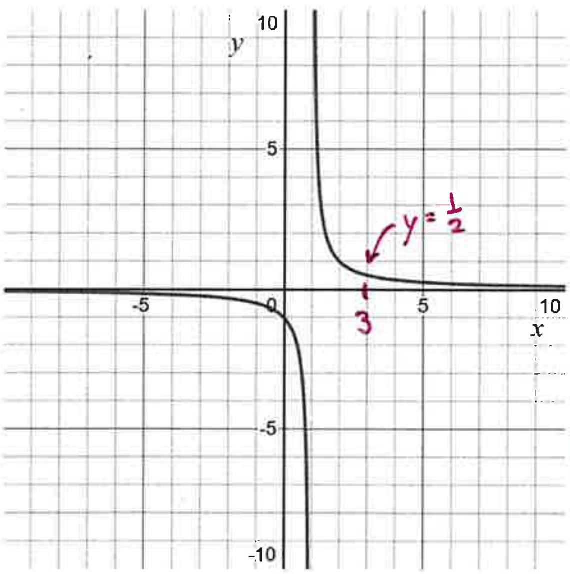
Consider the behaviour of the function

$$f(x) = \frac{x-3}{x^2-4x+3}$$

When x is near 3. The following table gives values of x approaching 3 (but not equal to 3)

Approaching from the Left		**Approaching from the Right**	
$x < 3$	$f(x)$	$x > 3$	$f(x)$
2.5	0.666 667	3.5	0.400 000
2.9	0.526 316	3.1	0.476 190
2.99	0.502 513	3.01	0.497 512
2.999	0.500 250	3.001	0.499 750
2.9999	0.500 025	3.0001	0.499 975

Looking at the graph of this function you can see that the same information is borne out that as x approaches 3, the value of the function approaches $\frac{1}{2}$.



As $x \to 3$ from the left
notation 3^- $y \to \frac{1}{2}$

As $x \to$ from the right
notation 3^+ $y \to \frac{1}{2}$

consider as $x \to 1$ from left and right
look at the graph:
$x \to 1^-$ $y \to -\infty$
$x \to 1^+$ $y \to \infty$
since different — Limit Does Not Exist

Mathematically we say, "the limit of $\frac{x-3}{x^2-4x+3}$ as x approaches 3 is equal to $\frac{1}{2}$." Which can be written as

$$* \quad \lim_{x\to 3}\frac{x-3}{x^2-4x+3} = \frac{1}{2}$$

Happens to be value of the function too, not always the case.

For a function $f(x)$, if we allow x to become sufficiently close to a, but not equal to a and the value of $f(x)$ becomes closer to a value L, then we say the limit of $f(x)$ as x approaches a, equals L.

$$\lim_{x\to a} f(x) = L$$

Approach from left and the right

* If we just plug 3 in we get: $\frac{3-3}{3^2-4(3)+3} = \frac{0}{0}$ ← this is why we need limits

Ex. 1
Find $\lim_{x\to 5}(x^2 + 2x - 3)$.

quadratic, continuous everywhere

in this case we can just plug in:

$5^2 + 2(5) - 3$

$25 + 10 - 3$

$\boxed{32}$

* As you'll see in a moment, we can always compute individual terms too

$\lim_{x\to 5} x^2 = 25$ $\lim_{x\to 5} -3 = -3$

$\lim_{x\to 5} 2x = 10$

$25 + 10 - 3$

$\boxed{32}$

Properties of Limits

Suppose the following two limits both exist and c is a constant.

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

If that is the case, then the following properties of limits are also true.

	Mathematical Statement	Stated in Words
1.	$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$	The limit of a sum is the sum of the limits.
2.	$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$	The limit of a difference is the difference of the limits.
3.	$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$	The limit of a constant times a function is the constant times the limit of the function.
4.	$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$	The limit of a product is the product of the limits.
5.	$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$	The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0).
6.	$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ if n is a positive integer	The limit of a power is the power of the limit.
7.	$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ if the root on the right side exists	The limit of a root is the root of the limit (if the root exists).

Starting with the basic limits

$\lim_{x\to a} x = a$	$\lim_{x\to a} c = c$	(c is a constant)

Then from properties 6 and 7 you can deduce the following:

$\lim_{x\to a} x^n = a^n$	$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$	(if $\sqrt[n]{a}$ exists)

Ex. 2
Find $\lim_{x \to 5}(x^2 + 2x - 3)$

Using properties of limits

$\lim_{x \to 5} x^2 + \lim_{x \to 5} 2x + \lim_{x \to 5} -3$

Properties #1 and 2

$5^2 + 2(5) - 3$

$25 + 10 - 3 \rightarrow \boxed{32}$

Polynomial and Rational Functions

Recall that a **polynomial** is a function of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where $a_0, a_1, \ldots, a_n$ are constants. Also recall that a **rational function** is a ratio of polynomials. The properties of limits can be used to show whether a function is *continuous* or not.

(a)	Any polynomial P is continuous at every number; that is, $\lim_{x \to a} P(x) = P(a)$ Direct Substitution
(b)	Any rational function $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials, is continuous at every number a such that $Q(a) \neq 0$; that is, $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}, Q(a) \neq 0$ Direct Substitution

Redo **Ex. 2** using property (a)

Again, direct substitution is the best way to compute a limit

$\lim_{x \to 5} (x^2 + 2x - 3) \rightarrow 5^2 + 2(5) - 3$

$\boxed{32}$

Both function continuous for these x-values so we can use Direct Substitution

Ex. 3
Evaluate using the properties of limits.

(a) $\lim_{x \to 1} \frac{x^4 - 5x^2 + 1}{x + 2}$

$\frac{1^4 - 5(1)^2 + 1}{1 + 2}$

$\frac{1 - 5 + 1}{3}$

$\frac{-3}{3}$

$\boxed{= -1}$

(b) $\lim_{x \to 3} \sqrt{x^2 + x}$

$\sqrt{3^2 + 3}$

$\sqrt{9 + 3}$

$\sqrt{12}$ ← simplify the radical

$\sqrt{4 \cdot 3}$

$\sqrt{4}\sqrt{3}$

$\boxed{2\sqrt{3}}$

If the following is true then we say the function is **continuous at *a*.**

$$\lim_{x \to a} f(x) = f(a)$$

Ex. 4
Evaluate the following limit.

$$\lim_{x\to 4}\frac{x^2-16}{x-4}$$

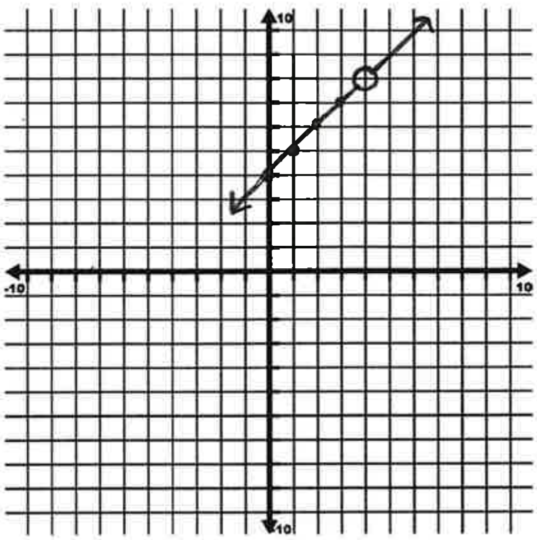
Direct substitution gives $\frac{0}{0}$

world explodes

Factor and see is discontinuity is removed

$$\lim_{x\to 4}\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}$$

$$\lim_{x\to 4}(x+4) \rightarrow 4+4 = \boxed{8}$$



$$f(x)=\frac{x^2-16}{x-4} \quad x\neq 4$$

$$=\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}$$ } cancelling creates a Hole

when $x=4$

$f(4)=8$

Ex. 5
Evaluate the following limit.

$$\lim_{x\to 2}\frac{x^3-8}{x^2-3x+2}$$ ← Diff of cubes

Direct substitution gives $\frac{0}{0}$ try factoring

$$\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x-1)}$$

Now try Direct Sub.

$$\lim_{x\to 2}\frac{2^2+2(2)+4}{2-1} = \frac{12}{1} = \boxed{12}$$

Ex. 6
Evaluate the following limit.

FOIL

$$\lim_{h\to 0}\frac{(2+h)^2-4}{h}$$

* If factoring doesn't work try expanding

Direct sub gives

$\frac{0}{0}$

when we factor

$$\frac{[(2+h)+2][(2+h)-2]}{h}$$

$$\frac{[4][0]}{0}=\frac{0}{0}$$

so try

$$\frac{(2+h)(2+h)-4}{h} \rightarrow \frac{4+4h+h^2-4}{h}$$

$$\frac{h^2+4h}{h} \rightarrow \frac{\cancel{h}(h+4)}{\cancel{h}}$$

$$\lim_{h\to 0} h+4 = \boxed{4}$$

Ex. 7
Evaluate the following limit.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

Direct sub gives

$\frac{0}{0}$

$$\frac{(\sqrt{x+1}-1)}{x}\cdot\frac{(\sqrt{x+1}+1)}{\sqrt{x+1}+1}$$

$$\lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$$

$$\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} = \boxed{\frac{1}{2}}$$

* Radicals
try radicalizing

* If you have fractions in the numerator of a complex fractions
try adding/subtracting them.

See # 5c in workbook

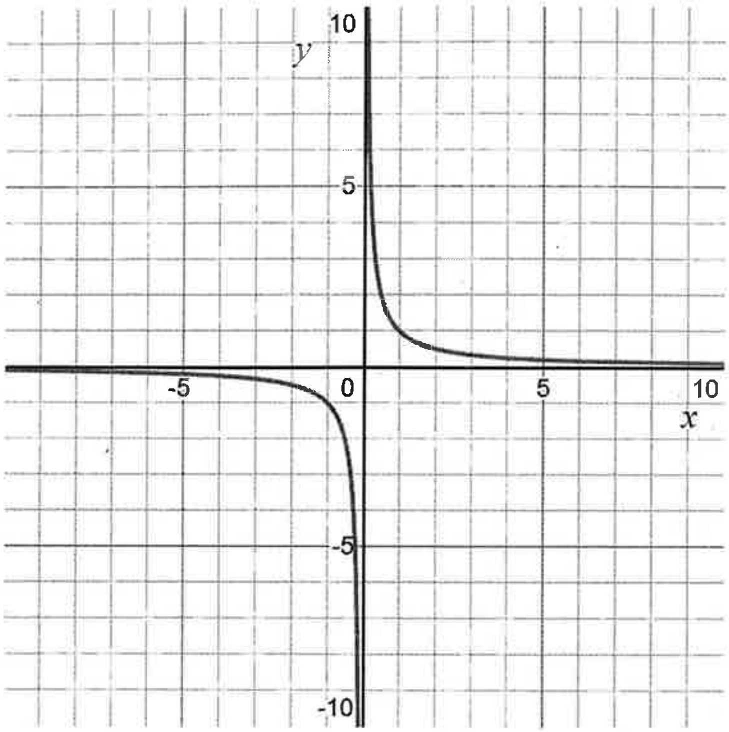
Ex. 8
Show that the following limit does not exist.

$$\lim_{x\to 0}\frac{1}{x}$$

Direct sub gives $\frac{1}{0}$ which DNE

From right

$\lim_{x\to 0^+}\frac{1}{x}$ $y \to +\infty$



From the left

$\lim_{x\to 0^-}\frac{1}{x}$ $y \to -\infty$

So $\lim_{x\to 0}\frac{1}{x}$

DOES NOT EXIST

Because ∞ is not a number

and because

Left Sided Limit $\neq$ Right Sided Limit

Homework Assignment
Exercise 1.2: #1, 2, 3adfg, 4aceg, 5ace, 6adej, 7, 10