11.4 Integration by Parts

As with the previous section where we reversed the Chain Rule to develop the Substitution Rule for integration, other rules of differentiation have a corresponding rule for integration. In this section we will reverse the Product Rule to develop the rule for integration by parts.

Recall that the Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

In terms of integrals, this becomes

$$\int [f'(x)g(x) + g'(x)f(x)]dx = f(x)g(x)$$

or written in another way

$$\int f'(x)g(x)dx + \int g'(x)f(x)dx = f(x)g(x)$$

We can rewrite this equation as follows.

Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Ex. 1

Find

$$\int xe^x\,dx$$

Let $x = f(x)$ $f'(x) = 1$

$e^x = g(x)$ $g'(x) = e^x dx$

$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)dx$

$= xe^x - \int 1\cdot e^x dx$

$= xe^x - e^x + C$

$= \boxed{e^x(x-1) + C}$

The goal of integration by parts is to get a simpler integral than the one we started with. The choice is somewhat arbitrary, but there can be better choices that will better simplify the problem. Another way $\int xe^x dx$ from Example 1 could have been evaluated would be to have chosen $f(x) = e^x$ and $g'(x) = x$, then $f'(x) = e^x$ and $g(x) = \frac{1}{2}x^2$, so integration by parts would give

$$\int xe^x dx = e^x\left(\frac{1}{2}x^2\right) - \frac{1}{2}\int x^2e^x dx$$

But $\int x^2 e^x dx$ is a more difficult integral than the one we started with. In general, we try to choose f to be a function that becomes *simpler* when differentiated, as long as $g'(x)$ can be readily integrated to give $g(x)$.

Integration by parts is often stated in differential notation rather than function notation. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. Substituting these into the previous function statement of integration by parts we get the following more simplified version.

Integration by Parts in Differential Notation

$$\int u\, dv = uv - \int v\, du$$

Using the above formula, we could rewrite the solution to Example 1 as follows:

Let $u = x$ and $v = e^x$. Then $du = dx$ and $dv = e^x dx$. Then using integration by parts, we have

$$\int xe^x dx = xe^x - \int e^x dx \boxed{= xe^x - e^x + C}$$

Ex. 2

Find:

$$\int x \cos 3x \, dx$$

Let $x = u$ $dv = \cos 3x\, dx$

$dx = du$ $v = \frac{1}{3}\sin 3x$

$$\int u\,dv = uv - \int v\,du$$

$$= x \cdot \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x \, dx$$

$$= x \cdot \frac{1}{3}\sin 3x - \left(-\frac{1}{9}\cos 3x\right)$$

$$= \boxed{\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C}$$

Ex. 3

Evaluate:

$$\int x^2 \sin 3x \, dx$$

can call $\frac{2}{3}C = K$

Let $u = x^2$ $dv = \sin 3x$

$du = 2x\,dx$ $v = -\frac{1}{3}\cos 3x$

$$\int u\,dv = uv - \int v\,du$$

$$= x^2 \cdot -\frac{1}{3}\cos 3x - \int -\frac{1}{3}\cos 3x \cdot 2x\,dx$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx$$

from Ex. 2

$$-\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left(\frac{1}{3}\sin 3x + \frac{1}{9}\cos 3x + C\right)$$

$$= \boxed{-\frac{1}{3}x^2\cos 3x + \frac{2}{9}\sin 3x + \frac{2}{27}\cos 3x + \frac{2}{3}C}$$

Ex. 4

Find:

$$\int x^2 \ln x \, dx$$

let $u = \ln x \quad dv = x^2$

$du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

$\int u\,dv = uv - \int v\,du$

$\int x^2 \ln x\,dx = \ln x\left(\frac{1}{3}x^3\right) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx \longrightarrow \frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 + C$

$\boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$

Definite Integration by Parts

When using integration by parts to find a definite integral, the process is essentially the same as with indefinite integrals, except you evaluate both sides of the formula between the appropriate limits. Written in function notation definite integration by parts looks like the following statement.

Definite Integration by Parts

$$\int_a^b f(x)\,g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

Ex. 5

Evaluate the following integral.

$$\int_1^e \ln x\,dx$$

let $u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$\ln x(x)\Big]_1^e - \int_1^e x\cdot\frac{1}{x}dx = x\ln x\Big]_1^e - \int_1^e 1\,dx$

$\left[e\ln e - \ln 1\right] - x\Big]_1^e$

$\left[e\ln e - \ln 1\right] - \left[e - 1\right]$

$\Rightarrow e - 0 - [e-1]$

$= e - e + 1$

$= \boxed{1}$

Homework Assignment

- Practice Problems: #1-4