11.3 The Substitution Rule

Consider using the chain rule to differentiate $y = \sqrt{x^2+1}$, we would get the following as a result.

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) \boxed{= \frac{x}{\sqrt{x^2+1}}}$$

Rewriting this in terms of indefinite integrals, we have

$$\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1} + C$$

This process of thinking of the Chain Rule in reverse can be used to evaluate other integrals. Recall the general version of the Chain Rule:

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

Reversing this and rewriting in terms of integrals, we get

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

Another way to approach this is to change from using the variable x to another variable $u = g(x)$. This replacement can simplify the look of the problem.

$$\int F'(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int F'(u)du$$

Finally, if we now write $F' = f$, we get

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Ex. 1

Find:

$\int (x^2 - 5)^8 2x\, dx$

Let

$g(x) = u = x^2 - 5$ then $f(u) = u^8$

and $g'(x) = 2x\,dx$ $f(g(x)) = (x^2-5)^8$

$\int (x^2-5)^8 2x\,dx = \int f(g(x))\,g'(x)\,dx$

$= \int f(u)\,du$

$= \int u^8\,du$

$\frac{1}{9}u^9 + C = \boxed{\frac{(x^2-5)^9}{9} + C}$

One easy way to remember the Substitution Rule is through the concept of a *differential*. If $u = g(x)$ is a differentiable function, we can define the **differential dx** to be an independent variable; that is, dx can be given the value of any real number. Then the **differential du** is defined in terms of dx by the equation

$$du = g'(x)dx$$

For example, if $u = 5x^6$, then $du = 30x^5 dx$.

Regarding the dx and the du after the integral signs as differentials, we have

$$\int f(g(x))g'(x)dx = \int f(u)du$$

which is the Substitution Rule. This means that in using the Substitution Rule, all we have to do is treat dx and du after the integral signs as differentials. For instance, Example 1 could be solved as follows:

Ex. 1

Solution 2

Let $u = x^2 - 5$. Then $du = 2x\,dx$, so

$$\int (x^2 - 5)^8 2x\,dx = \int u^8\,du$$
$$= \frac{u^9}{9} + C$$
$$\boxed{= \frac{(x^2-5)^9}{9} + C}$$

Ex. 2

Evaluate the following integral.

$$\int \frac{x^2}{\sqrt{1-x^3}}\,dx$$

Let $u = 1-x^3$

$du = -3x^2$ we have x^2 but need $-3x^2 = du \rightarrow x^2 = -\frac{1}{3}du$

$$\int \frac{-\frac{1}{3}du}{\sqrt{u}} = -\frac{1}{3}\int \frac{du}{\sqrt{u}} = -\frac{1}{3}\int u^{-\frac{1}{2}}du \rightarrow -\frac{1}{3}\cdot 2u^{\frac{1}{2}} + C$$

$$\Rightarrow -\frac{2}{3}u^{\frac{1}{2}} + C = -\frac{2}{3}(1-x^3)^{\frac{1}{2}} + C$$

In trying to choose an appropriate substitution, try to choose u to be some function in the integrand whose differential also appears (except for perhaps a constant factor). In Example 1, we chose $u = x^2 - 5$ because $du = 2x\,dx$ also appears. In Example 2, we chose $u = 1 - x^3$ because $du = -3x^2\,dx$ also appears.

Ex. 3

Find:

$$\int \frac{\ln x}{x}\,dx$$

Let $\ln x = u$

$\frac{1}{x}dx = du$

then $\int u\,du \rightarrow \frac{1}{2}u^2 = \boxed{\frac{(\ln x)^2}{2} + C}$

Ex. 4

Find:
$$\int \sin 4x\,dx$$

Let $u = 4x$

$du = 4dx \rightarrow \frac{1}{4}du = dx$

$$\frac{1}{4}\int \sin u\,du \rightarrow \frac{1}{4}(-\cos u) + C$$

$$\rightarrow \boxed{-\frac{1}{4}\cos 4x + C}$$

Ex. 5

Find:
$$\int (2+\sin x)^{10}\cos x\; dx$$

Let $u = 2 + \sin x$

$du = \cos x\,dx$

$$\int u^{10}du \rightarrow \frac{1}{11}u^{11} + C$$

$$= \boxed{\frac{(2+\sin x)^{11}}{11} + C}$$

The substitution rule can also be used for definite integrals as well. All we have to do is to change the limits of integration for the appropriate values of u.

Substitution Rule for Definite Integrals
If $u = g(x)$, then
$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Ex. 6

Evaluate the following definite integral.

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2\,du = \frac{1}{\sqrt{x}}dx$

$\int_1^9 e^u \, 2du$ but

$u = \sqrt{x}$

so boundaries become $\sqrt{9} = 3$

$\sqrt{1} = 1$

$2\int_1^3 e^u du \rightarrow 2e^u\Big]_1^3 \rightarrow 2(e^3 - e^1) \rightarrow \boxed{2(e^3 - e)}$

Ex. 7

Find the area under the curve $y = \frac{1}{2x+1}$ from 0 to 1.

Let $2x+1 = u$

$2 = du$

$1 = \frac{1}{2}du$

if $u = 2x+1$ our boundaries change to

$2(0)+1 = 1$

$2(1)+1 = 3$

$y = \frac{1}{u}\cdot\frac{1}{2}du$

$= \frac{1}{2}\int \frac{1}{u}du \rightarrow \frac{1}{2}\ln u \rightarrow \frac{1}{2}\left(\ln u\right]_1^3$

$\frac{1}{2}(\ln 3 - \ln 1)$

$= \boxed{\frac{1}{2}\ln 3}$

Homework Assignment

- Practice Problems: #1, 2, 3odd, 4abcd, 6