11.2 The Fundamental Theorem of Calculus

In the previous section we used the definition of an integral to calculate integrals as limits of sums. This approach can sometimes be long and difficult so we will now study the Fundamental Theorem of Calculus, which will provide us with a much more streamlined approach to calculating integrals.

For the special case where $f(x) \geq 0$, we know the definite integral $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$ from a to b. From the previous unit we also know that the area is equal to $F(b) - F(a)$, where F is any antiderivative of f. Therefore, in this case, we have

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{where } F' = f$$

It turns out that the equation above is true even when f is not a positive function.

Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is an antiderivative of f.

Ex. 1

Evaluate:

$$\int_{-1}^2 x^3 dx = F(2) - F(-1)$$

$$= \frac{1}{4}x^4 \Big]_{-1}^{2} \rightarrow \frac{1}{4}(2)^4 - \frac{1}{4}(-1)^4$$

$$\rightarrow \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$$

When compared to the solution from Example 3 in the previous section, you can appreciate how the Fundamental Theorem of Calculus gives a much simpler solution. The following notation is used.

$$F(b) - F(a) = F(x)\Big]_a^b \qquad \text{or } F(x)\Big|_a^b$$

The conclusion of the Fundamental Theorem of Calculus can be written as the following.

$$\int_a^b f(x)dx = F(x)\Big]_a^b \qquad \text{where } F' = f$$

Using the theorem, the solution to Example 1 could be streamlined as follows:

$$\int_{-1}^{2} x^3 dx = \frac{x^4}{4}\Bigg]_{-1}^{2} = \frac{2^4-(-1)^4}{4} = \boxed{\frac{15}{4}}$$

Ex. 2

Find:

$$\int_{\pi}^{2\pi} \sin x \, dx$$

Antiderivative of $\sin x = -\cos x$

$$-\cos x \Big]_{\pi}^{2\pi} \rightarrow -\cos 2\pi - (-\cos \pi)$$

$$= -1 + (-1) = \boxed{-2}$$

The symbol $\int f(x)dx$ represents the antiderivative of f, more specifically it represents the **indefinite integral of f** because the constant C is not specified. The table below lists many common indefinite integrals.

Table of Indefinite Integrals (Antiderivatives)	
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\int \sin x \, dx = -\cos x + C$	$\int \cos x \, dx = \sin x + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \csc^2 x \, dx = -\cot x + C$
$\int \sec x \tan x \, dx = \sec x + C$	$\int \csc x \cot x \, dx = -\csc x + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a \neq 1)$
$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

Any formula in the table can be verified by differentiating the function on the right side.

It is important to distinguish between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function. The Fundamental Theorem of Calculus gives the connection between the two.

$$\int_a^b f(x)dx = \int f(x)dx \Big]_a^b$$

The following rules can be used to find antiderivatives of more complicated functions.

Properties of Indefinite Integrals

$$\int cf(x)dx = c\int f(x)dx \quad (c \text{ is a constant})$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

And it follows from the Fundamental Theorem of Calculus that the rules for definite integrals are the same.

Properties of Definite Integrals

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad (c \text{ is a constant})$$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)$$

Ex. 3

Find the general indefinite integral:

$$\int (6x^2 + \csc^2 x)dx \rightarrow 6\int x^2 dx + \int \csc^2 x\, dx$$

$$= 6\,\frac{1}{3}x^3 + (-\cot x)$$

$$= 2x^3 - \cot x + C$$

Ex. 4

Find:

$$\int_2^5 (2x^3 - 3x^2 + 7x + 2)\, dx$$

$$\rightarrow \frac{2x^4}{4} - \frac{3x^3}{3} + \frac{7x^2}{2} + 2x \Big]_2^5$$

$$\frac{5^4}{2} - 5^3 + \frac{7(25)}{2} + 10 - \left[\frac{16}{2} - 2^3 + \frac{7(4)}{2} + 4\right]$$

$$285 - 18$$

$$= \boxed{267}$$

Ex. 5

Evaluate:

$$\int_1^8 \frac{1}{\sqrt[3]{x^2}}dx \rightarrow \int_1^8 x^{-2/3}\,dx$$

$$\rightarrow \frac{1}{-\frac{2}{3}+1}x^{\frac{1}{3}} = 3x^{\frac{1}{3}}\Big]_1^8$$

$$3(8)^{\frac{1}{3}} - 3(1)^{\frac{1}{3}}$$

$$\rightarrow 6-3$$

$$\boxed{3}$$

Ex. 6

Find:

$$\int_1^4 \frac{t^2+\sqrt{t}-2}{t}dt \longrightarrow \int_1^4\left(\frac{t^2}{t}+\frac{\sqrt{t}}{t}-\frac{2}{t}\right)dt$$

$$\rightarrow \int_1^4\left(t+t^{-\frac{1}{2}}-\frac{2}{t}\right)dt$$

$$\rightarrow \frac{t^2}{2}+2t^{\frac{1}{2}}-2\ln t\Big]_1^4$$

$$\frac{16}{2}+4-2\ln 4-\left(\frac{1}{2}+2-2\ln 1\right)$$

$$\Rightarrow 8+4-2\ln 4-\frac{1}{2}-2+2$$

$$\boxed{9.5-2\ln 4}$$

Ex. 7

Find:

$$\int_0^1 \frac{1}{x^2+1}dx$$

Inverse Tan, you'll see this in University

$\tan^{-1}\Big]_0^1$

$\tan^{-1} 1 - \tan^{-1} 0$

$\frac{\pi}{4} - 0$

$\boxed{\frac{\pi}{4}}$

The Fundamental Theorem of Calculus says that if f is continuous then the following is true.

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F' = f$$

Now if we replace f with F' on the left side of the above equation we can rewrite it as the following equation.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This statement says that when we take a function F, differentiate it, then integrate that result, we arrive at an expression that involves the original function. What this means is that the Fundamental Theorem of Calculus says that differentiation and integration are inverse processes. This relationship was first noticed by Isaac Barrow and was used by Newton and Leibniz to make calculus the powerful tool it is today for solving problems in mathematics and science.

Homework Assignment

- Practice Problems: #1, 2