Section 11.1 – Practice Problems

1.

a) Evaluate the following integral

$\Delta x = \frac{4-0}{n} = \frac{4}{n}$ $x_i = a + i\Delta x = 0 + \frac{4i}{n}$

$$\int_0^4 (x^2 - 2x)dx$$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(\left(\frac{4i}{n}\right)^2 - 2\left(\frac{4i}{n}\right)\right) \to \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(\frac{16i^2}{n^2} - \frac{8i}{n}\right)$

$\to \lim_{n\to\infty} \sum_{i=1}^{n} \frac{64i^2}{n^3} - \frac{32i}{n^2} = \lim_{n\to\infty} \left[\frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{32}{n^2}\sum_{i=1}^{n} i\right] \Rightarrow \lim_{n\to\infty} \frac{64}{n^3}\frac{(n(n+1)(2n+1))}{6} - \frac{32}{n^2}\frac{(n(n+1))}{2}$

$\Rightarrow \lim_{n\to\infty} \frac{64(2n^2+3n+1)}{n^2 6} - \frac{32(n^2+n)}{2n^2} \Rightarrow \frac{128n^2}{6n^2} + \frac{192n}{6n^2} + \frac{64}{6n^2} - \frac{32n^2}{2n^2} - \frac{32n}{2n^2} = \frac{128}{6} - \frac{32}{2} = \boxed{16/3}$

b) Find the value of the approximating sum

$\Delta x = \frac{4+0}{8} = \frac{1}{2}$

$$\sum_{i=1}^{8} f(x_i)\Delta x$$

$x_i = a + i\Delta x$
$0 + \frac{1}{2}i$

where $f(x) = x^2 - 2x, a = 0$ and $b = 4$

$\sum_{i=1}^{8} f(x_i)\Delta x = \frac{1}{2}\sum_{i=1}^{8} f(\tfrac{1}{2}i) = \frac{1}{2}\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4)\right]$

$= \frac{1}{2}\left[-0.75 - 1 - 0.75 + 0 + 1.25 + 3 + 5.25 + 8\right] = \boxed{7.5}$

2. Evaluate each integral.

a)

$\Delta x = \frac{5}{n}$

$$\int_{-2}^{3} (1-4x)dx$$

$x_i = -2 + \frac{5i}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} (1 - 4(-2 + \frac{5i}{n}))\frac{5}{n} = \sum_{i=1}^{n} (9 - \frac{20i}{n})\frac{5}{n}$

$= \frac{5}{n}\left[9\sum_{i=1}^{n} 1 - \frac{20}{n}\sum_{i=1}^{n} i\right] = \frac{45}{n}\cdot n - \frac{100}{n^2}\frac{(n(n+1))}{2}$

$\lim_{n\to\infty} 45 - \frac{50n^2}{n^2} - \frac{50n}{n^2} \to 45 - 50 = \boxed{-5}$

b)

$\Delta x = \frac{1}{n}$

$$\int_0^1 (1 + 4x - 6x^2)dx$$

$x_i = \frac{i}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} (1 + \frac{4i}{n} - \frac{6i^2}{n^2})\frac{1}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} 1 + \frac{4}{n}\sum_{i=1}^{n} i - \frac{6}{n^2}\sum_{i=1}^{n} i^2\right]$

$= \frac{1}{n}\left[n + \frac{4}{n}\frac{(n(n+1))}{2} - \frac{6}{n^2}\frac{(n(n+1)(2n+1))}{6}\right]$

$= \frac{1}{n}\left[n + 2n + 2 - \frac{(2n^2+3n+1)}{n}\right] \to \frac{1}{n}\left[3n + 2 - 2n - 3 - \frac{1}{n}\right]$

$\to 3 + \frac{2}{n} - 2 - \frac{3}{n} - \frac{1}{n^2} \to \lim_{n\to\infty} \boxed{1}$

c) $\Delta x = \frac{1}{n}$

$x_i = \frac{i}{n}$ $\qquad \int_0^1 x^3dx$

$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^3 \to \frac{1}{n}\sum_{i=1}^{n}\frac{i^3}{n^3} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$

$\to \lim_{n\to\infty}\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{1}{n^2}\cdot\frac{n^2+2n+1}{4}$

$= \frac{n^2}{4n^2}+\frac{2n}{4n^2}+\frac{1}{4n^2} \to \lim_{n\to\infty}\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}$

$= \boxed{\frac{1}{4}}$

d)

$\Delta x = \frac{3}{n}$ $\qquad \int_1^4 (x^2-6)dx$

$x_i = 1+\frac{3i}{n}$

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\left(1+\frac{3i}{n}\right)^2-6\right) \to \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}-6\right)$

$\frac{3}{n}\sum_{i=1}^{n}\left(-5+\frac{6i}{n}+\frac{9i^2}{n^2}\right) \to \lim_{n\to\infty}\frac{3}{n}\left[-5\sum_{i=1}^{n}1+\frac{6}{n}\sum_{i=1}^{n}i+\frac{9}{n^2}\sum_{i=1}^{n}i^2\right]$

$\to \frac{3}{n}\left[-5n+\frac{6}{n}\frac{(n(n+1))}{2}+\frac{9}{n^2}\frac{(n(n+1)(2n+1))}{6}\right]$ $\qquad \boxed{3}$

$\to -15+\frac{18}{n^2}\frac{(n^2+n)}{2}+\frac{27}{n^3}\frac{(n(2n^2+3n+1))}{6}$

$\to -15+9+\frac{9}{n}+\frac{9}{n^2}\left(\frac{2n^2+3n+1}{2}\right) \to -15+9+\frac{9}{n}+9+\frac{27}{2n}+\frac{9}{2n^2}$

3. Evaluate each integral and interpret the value as a difference of areas.

a) $\Delta x = \frac{3}{n}$ $\quad x_i = \frac{3i}{n}$ $\quad \int_0^3 (1-x^2)dx$

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(1-\frac{9i^2}{n^2}\right) \to \frac{3}{n}\left[\sum_{i=1}^{n}1-\frac{9}{n^2}\sum_{i=1}^{n}i^2\right]$

$\to \frac{3}{n}\left[n-\frac{9}{n^2}\frac{(n(n+1)(2n+1))}{6}\right] \to \frac{3}{n}\left[n-\frac{9}{n}\frac{(2n^2+3n+1)}{6}\right]$

$\frac{3}{n}\left[n-\frac{18n^2}{6n}-\frac{27n}{6n}-\frac{9}{6n}\right] \to \frac{3n}{n}-\frac{9n^2}{n^2}-\frac{27n}{2n^2}-\frac{27}{6n^2}$

$\lim_{n\to\infty} 3-9-\frac{27}{2n}-\frac{27}{6n^2} = \boxed{-6}$ USE DESMOS CONSIDER GRAPH

b) $\Delta x = \frac{2}{n}$ $\quad \int_3^5 (2x-7)dx$ $\quad \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(2\left(3+\frac{2i}{n}\right)-7\right)$

$x_i = 3+\frac{2i}{n}$

$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}-1+\frac{4i}{n} \to \frac{2}{n}\left[-1\sum_{i=1}^{n}1+\frac{4}{n}\sum_{i=1}^{n}i\right]$

$\lim_{n\to\infty}\frac{2}{n}\left[-1n+\frac{4}{n}\frac{(n(n+1))}{2}\right] = \frac{2}{n}\left[-n+2n+2\right]$

$\lim_{n\to\infty} -2+4+\frac{4}{n} = \boxed{2}$ USE DESMOS CONSIDER GRAPH.

4. Evaluate the following.

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a+\frac{(b-a)i}{n}$ $\quad \int_a^b x^2dx$

$\lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left(a+\frac{(b-a)i}{n}\right)^2 \to \frac{(b-a)}{n}\sum_{i=1}^{n}a^2+\frac{2a(b-a)i}{n}+\left(\frac{(b-a)i}{n}\right)^2$

$\lim_{n\to\infty}\frac{(b-a)}{n}\left[a^2\sum_{i=1}^{n}1+\frac{2a(b-a)}{n}\sum_{i=1}^{n}i+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i^2\right] \to \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left[a^2n+\frac{2a(b-a)}{n}\cdot\frac{n(n+1)}{2}+\frac{(b-a)^2}{n^2}\frac{(n(n+1)(2n+1))}{6}\right]$

$\lim_{n\to\infty}\frac{(b-a)}{n}\left[a^2n+a(b-a)(n+1)+(b-a)^2\frac{(n+1)(2n+1)}{6n}\right] \to a^2(b-a)+\frac{a(b-a)^2(n+1)}{n}+(b-a)^3\frac{(n+1)(2n+1)}{6n^2}$

$\lim_{n\to\infty} a^2(b-a)+a(b-a)^2\left(1+\frac{1}{n}\right)+\frac{(b-a)^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3} \longrightarrow a^2b-\cancel{a^3}+ab^2-2a^2b+\cancel{a^3}+\frac{(b-a)^3}{3}$

$ab^2-a^2b+\frac{1}{3}(b-a)^3$

$\to ab^2-a^2b+\frac{1}{3}(b^3-3b^2a+3a^2b-a^3) \to ab^2-a^2b+\frac{1}{3}b^3-b^2a+a^2b-\frac{1}{3}a^3 \to \boxed{\frac{1}{3}(b^3-a^3)}$