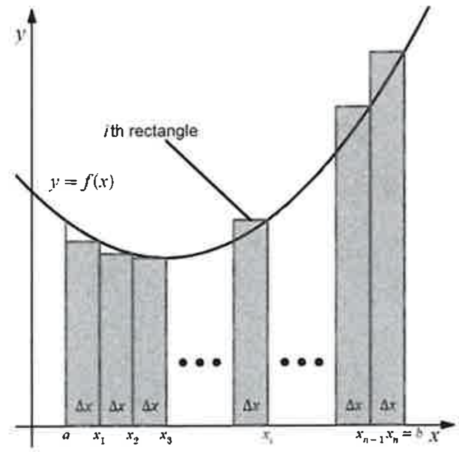
11.1 The Definite Integral

In this unit we will be studying the other main branch of calculus, integral calculus. Using antiderivatives, we will be able to compute integrals to calculate areas. We will also see how the Fundamental Theorem of Calculus links differential and integral calculus.

In the previous unit we saw that the area under a curve from a to b can be calculated using the following formula.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.



The kind of limit of a sum as shown above shows up frequently in calculating volumes, the lengths of curves, areas of surfaces, and physical quantities such as work and force. For this reason, we give this type of limit a special name and notation.

> Let f be a continuous function defined on an interval $[a, b]$. The **definite integral of f from a to b** is
>
> $$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$
>
> Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. The elongated S shape was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The dx refers to the variable that you are integrating with respect to. The procedure of calculating an integral is called **integration**. If the integrand is a positive function, then the integral represents an area, however this is not always the case as Example 1 illustrates.

> For the special case where $f(x) \geq 0$,
>
> $$\int_a^b f(x)dx = \text{the area under the graph of } f \text{ from } a \text{ to } b$$

Ex. 1

Evaluate the following integral.

upper limit

$$\int_0^5 (3x - x^2)dx$$

lower limit

$$\Delta x = \frac{5-0}{n} = \frac{5}{n}$$

$$x_i = a + i\Delta x \rightarrow 0 + \frac{5i}{n}$$

$$= \frac{5i}{n}$$

$$\int_0^5 (3x-x^2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\cdot\frac{5i}{n} - \left(\frac{5i}{n}\right)^2\right)\frac{5}{n} \rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{15i}{n} - \frac{25i}{n^2}\right)\frac{5}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{75i}{n^2} - \frac{125i^2}{n^3}$$

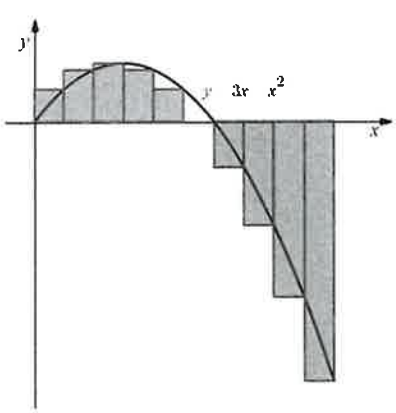
$$\rightarrow \frac{75}{n^2}\sum_{i=1}^{n} i - \frac{125}{n^3}\sum_{i=1}^{n} i^2$$

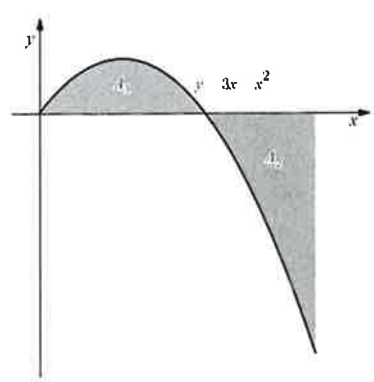
$$\rightarrow \frac{75}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{125}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\frac{5(n+1)}{2n} - \frac{125(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} \frac{75n+75}{2n} - \frac{125(2n^2+3n+1)}{6n^2}$$

$$\lim_{n\to\infty} \frac{75}{2} + \frac{75}{2n} - \frac{250}{6} - \frac{375}{6n} - \frac{125}{6n^2} \rightarrow \frac{75}{2} - \frac{250}{6} = \boxed{-\frac{25}{6}}$$

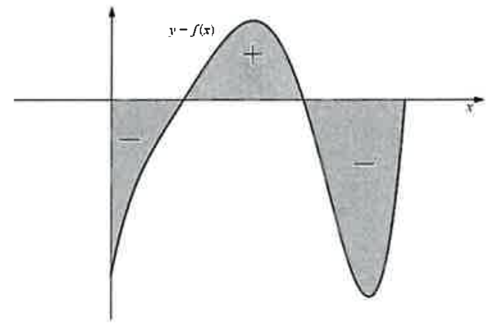




In general, if a function f takes on both positive and negative values, then $\int_a^b f(x)dx$ can be interpreted as a difference of areas.

$$\int_a^b f(x)dx = A_1 - A_2$$

Where A_1 is the area of the region *above* the x-axis and *below* the graph of f and A_2 is the area of the region *below* the x-axis and *above* the graph of f.



Ex. 3

Evaluate the following integral and interpret the value as a difference of areas.

$$\int_{-1}^{2} x^3 dx \qquad a=-1 \quad b=2 \qquad \Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(-1+\frac{3i}{n}\right)^3$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[-1+\frac{9i}{n} - \frac{27i^2}{n^2} + \frac{27i^3}{n^3}\right]\frac{3}{n} \rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} -\frac{3}{n} + \frac{27i}{n^2} - \frac{81i^2}{n^3} + \frac{81i^3}{n^4}$$

$$\lim_{n\to\infty} -\frac{3}{n}\sum_{i=1}^{n} 1 + \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{81}{n^4}\sum_{i=1}^{n} i^3$$

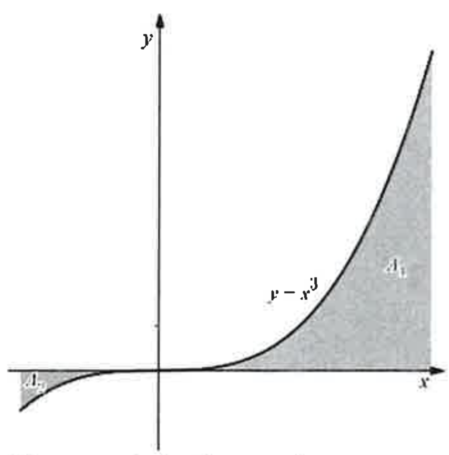
$$\lim_{n\to\infty} -\frac{3}{n}n + \frac{27}{n^2}\frac{n(n+1)}{2} - \frac{81}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4}\frac{n^2(n+1)^2}{4} \rightarrow -3 + \frac{27(n+1)}{2} - \frac{81(n+1)(2n+1)}{6n^2} + \frac{81(n+1)^2}{4n^2}$$

$$\rightarrow -3 + \frac{27n}{2n} + \frac{27}{2n} - \frac{81}{6}\left(\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}\right) + \frac{81}{4}\frac{(n^2+2n+1)}{n^2}$$

$$\rightarrow \lim_{n\to\infty} -3 + 27 + \frac{27}{2n} - \frac{81}{3} - \frac{81}{2n} - \frac{81}{6n^2} + \frac{81}{4} + \frac{162}{4n} + \frac{81}{4n^2}$$

$$\rightarrow -3 + \frac{27}{2} - \frac{81}{3} + \frac{81}{4}$$

$$= \boxed{\frac{15}{4}}$$



Homework Assignment

- Practice Problems: #1 – 4